1. a. Write $f(x)$ as powers of x as much as possible (remove denominators), so

$$f(x) = 6x^3 + 2x^{-2} - e^{2x}(x^2 - 9).$$

Apply power rules, product rule, and the rules for exponential yielding

$$\begin{aligned} f'(x) &= 6(3x^2) + 2(-2x^{-3}) - \left(e^{2x}(2x) + 2e^{2x}(x^2 - 9)\right) \\ &= 18x^2 - \frac{4}{x^3} - 2e^{2x}(x^2 + x - 9) \end{aligned}$$

b. Use the properties of logarithms to write

$$g(x) = 2e^{-3x} + 2\ln(x) - 5.$$

Use the rules of differentiation of exponentials and logarithms to give

$$\begin{aligned} g'(x) &= 2(-3)e^{-3x} + \frac{2}{x} + 0 \\ &= \frac{2}{x} - 6e^{-3x} \end{aligned}$$

c. Leave $h(x)$ in the form,

$$h(x) = 2x^6\ln(x) - e^{x^2+4x} + \frac{1}{2}e^{-4x}.$$

Apply power rules, product rule, chain rule, and the rules for exponentials and logarithms yielding

$$\begin{aligned} h'(x) &= 2\left((6x^5)\ln(x) + x^6\left(\frac{1}{x}\right)\right) - e^{x^2+4x}(2x+4) + \frac{-4}{2}e^{-4x} \\ &= 12x^5\ln(x) + 2x^5 - (2x+4)e^{x^2+4x} - 2e^{-4x} \end{aligned}$$

d. Write $k(t)$ as powers of t as much as possible, so

$$k(t) = \frac{1}{4}t^2 - 4t^{-\frac{1}{2}} + \frac{2 + e^{2t}}{t^2 - 3}.$$

Apply power rules, the rules for logarithms, and quotient rule yielding

$$\begin{aligned} k'(t) &= \left(\frac{2}{4}\right)t - 4\left(-\frac{1}{2}\right)t^{-\frac{3}{2}} + \frac{2(t^2-3)e^{2t} - 2t(2+e^{2t})}{(t^2-3)^2} \\ &= \frac{t}{2} + \frac{2}{\sqrt{t^3}} + \frac{2(t^2-3)e^{2t} - 2t(2+e^{2t})}{(t^2-3)^2} \end{aligned}$$

e. Write $p(w)$ as powers of w, so

$$p(w) = \frac{2\ln(w) + w}{w^3 - 8} - w^{2/5} + w^{-3}e^{-w}.$$

Apply power rules, product rule, and the rules for exponentials to give

$$\begin{aligned} p'(w) &= \frac{(w^3-8)(2/w+1)-3w^2(2\ln(w)+w)}{(w^3-8)^2} - \frac{2}{5}w^{-3/5} + \left(w^{-3}(-e^{-w}) - 3w^{-4}e^{-w}\right) \\ &= \frac{(w^3-8)(2/w+1)-3w^2(2\ln(w)+w)}{(w^3-8)^2} - \frac{2}{5}w^{-3/5} - \frac{e^{-w}}{w^3}\left(1+\frac{3}{w}\right) \end{aligned}$$

f. Write $q(z)$ as powers of z and use properties of logarithms, so

$$q(z) = 2Az\ln(z) - Bz^{1/2} + Cz^{-3}.$$

Apply power rules and the rules for logarithms to give

$$\begin{aligned} q'(z) &= 2A\left(\frac{z}{z} + \ln(z)\right) - \frac{B}{2}z^{-1/2} + C(-3)z^{-4} \\ &= 2A(1+\ln(z)) - \frac{B}{2\sqrt{z}} - \frac{3C}{z^4} \end{aligned}$$

g. Write $r(x)$ as follows:

$$r(x) = e^{2x}(x^3-5x+7)^4 - \frac{7x}{(x^2+2x+5)^{1/2}}.$$

Apply the product, quotient, and chain rule to obtain

$$\begin{aligned} r'(x) &= \left(e^{2x}4(x^3-5x+7)^3(3x^2-5) + 2e^{2x}(x^3-5x+7)^4\right) \\ &\quad - \frac{7(x^2+2x+5)^{1/2} - (7x/2)(x^2+2x+5)^{-1/2}(2x+2)}{(x^2+2x+5)} \\ &= 2e^{2x}(x^3-5x+7)^3(x^3+6x^2-5x-3) - \frac{7(x+5)}{(x^2+2x+5)^{3/2}} \end{aligned}$$

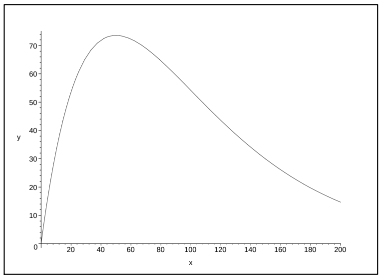
h. Leave $F(y)$ as

$$F(y) = (3y^2-4y+6)^5 + \ln(2y+9).$$

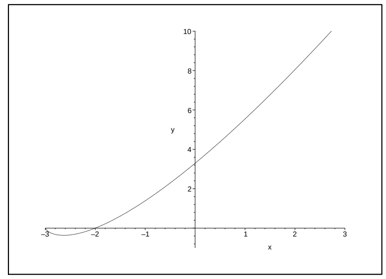
Apply the chain rule to obtain

$$F'(y) = 5(3y^2-4y+6)^4(6y-4) + \frac{2}{2y+9}$$

2. a. $y = 4xe^{-0.02x}$
Domain is all x.
y-intercept: $y(0) = 0$, so $(0, 0)$, which is also, the only x-intercept.
Horizontal asymptote: As $x \to \infty$, $y \to 0$, so $y = 0$ is a horizontal asymptote (looking to the right).
Derivative: By the product rule, $y'(x) = 4x(-0.02)e^{-0.02x} + 4e^{-0.02x} = 4e^{-0.02x}(1 - 0.02x)$
Critical points satisfy $y'(x) = 0$, so $1 - 0.02x = 0$ or $x = 50$. With $y(50) = 200e^{-1} \simeq 73.576$, $(50, 73.576)$ is a maximum.
Second derivative $y''(x) = 4e^{-0.02x}(-0.02) + 4(-0.02)e^{-0.02x}(1 - 0.02x) = -0.16(1 - 0.01x)e^{-0.02x}$.
Point of inflection ($y'' = 0$): At $x = 100$, $y(100) = 400e^{-2} \simeq 54.134$. Thus, $(100, 54.134)$.



Problem 2a

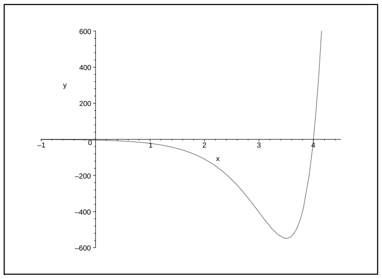


Problem 2b

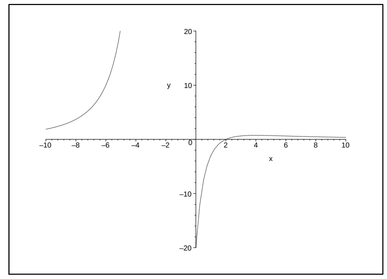
b. $y = (x + 3)\ln(x + 3)$
Domain is $x > -3$. The y-intercept is $3\ln(3) \simeq 3.2958$.
x-intercept: Where $(x + 3)\ln(x + 3) = 0$, which occurs when $\ln(x + 3) = 0$ or $x = -2$.
There are no asymptotes. (It can be shown that as $x \to -3$, $y \to 0$.)
Derivative: By the product rule, $y'(x) = \frac{x+3}{x+3} + \ln(x + 3) = 1 + \ln(x + 3)$.
Critical points satisfy $y'(x) = 0$, so $\ln(x+3) = -1$ or $x+3 = e^{-1} \simeq 0.3679$, so $x \simeq -2.6321$. When $x = e^{-1} - 3$, $y = -e^{-1}$ and is a minimum.
Second derivative $y''(x) = \frac{1}{x+3} > 0$ for $x > -3$. There is no point of inflection, and the function is concave up.

c. $y = (x - 4)e^{2x}$
Domain is all x.
y-intercept: $y(0) = -4$, so $(0, -4)$.
x-intercept: Since the exponential function is not zero, $y = 0$ when $x = 4$.
Horizontal asymptote: As $x \to -\infty$, $y \to 0$, so $y = 0$ is a horizontal asymptote (looking to the left).
Derivative: By the product rule, $y'(x) = 2(x - 4)e^{2x} + e^{2x} = (2x - 7)e^{2x}$.
Critical points satisfy $y'(x) = 0$, so $2x - 7 = 0$ or $x = 3.5$. With $y(3.5) = -0.5e^7 \simeq -548.3$, $(3.5, -548.3)$ is a minimum.
Second derivative $y''(x) = 2(2x - 7)e^{2x} + 2e^{2x} = 4(x - 3)e^{2x}$.
Point of inflection ($y'' = 0$): At $x = 3$, $y(3) = -e^6 \simeq -403.4$. Thus, $(3, -402.4)$.

d. $y = \dfrac{10(x - 2)}{(1 + 0.5x)^3}$
Domain is all $x \neq -2$.
y-intercept: $y(0) = -20$, so $(0, -20)$.
x-intercept: Numerator equal to zero, so $x = 2$ or $(2, 0)$
Vertical asymptote: $x = -2$.



Problem 2c



Problem 2d

Horizontal asymptote: The power of the denominator exceeds the power of the numerator, so $y = 0$ is a horizontal asymptote

Derivative: By the quotient rule, $y'(x) = 10\frac{(1+0.5x)^3-(x-2)3(1+0.5x)^2(0.5)}{(1+0.5x)^6} = \frac{10(4-x)}{(1+0.5x)^4}$.

Critical points satisfy $y'(x) = 0$, so $4 - x = 0$ or $x = 4$. With $y(4) = \frac{20}{27} \simeq 0.7407$, $(4, 0.7407)$ is a relative maximum.

Second derivative $y''(x) = 10\frac{-(1+0.5x)^4-(4-x)4(1+0.5x)^3(0.5)}{(1+0.5x)^8} = \frac{15(x-6)}{(1+0.5x)^5}$.

Since $y''(x) = 0$ and $x = 6$, there is a point of inflection at $(6, \frac{5}{8})$.

e. $y = \dfrac{2\,e^{2x}}{x-1}$

Domain is all $x \neq 1$.

y-intercept: $y(0) = -2$, so $(0, -2)$.

No x-intercept: The numerator is clearly never zero.

Vertical asymptote: $x = 1$.

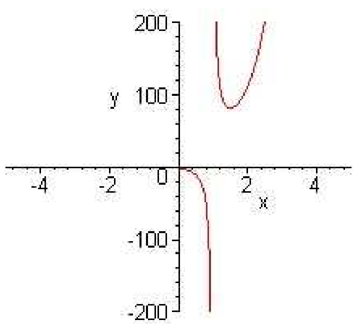
Horizontal asymptote: As $x \to -\infty$, $y \to 0$, so $y = 0$ is a horizontal asymptote (looking to the left).

Derivative: By the quotient rule, $y'(x) = \frac{2((x-1)2e^{2x}-e^{2x})}{(x-1)^2} = \frac{2e^{2x}(2x-3)}{(x-1)^2}$.

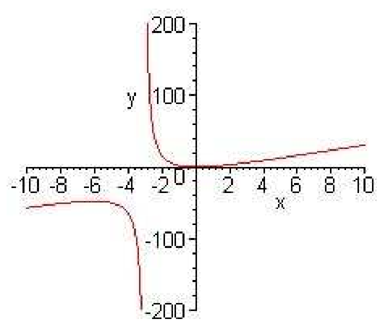
Critical points satisfy $y'(x) = 0$, so $2x-3 = 0$ or $x = 1.5$. With $y(1.5) = \frac{2e^3}{0.5} \simeq 80.342$, $(1.5, 80.342)$ is a minimum.

Second derivative $y''(x) = \frac{2((x^2-2x+1)(4x-4)e^{2x}-(2x-3)(2x-2)e^{2x})}{(x-1)^4} = \frac{4e^{2x}(2x^2-6x+5)}{(x-1)^3}$.

There are no points of inflection, since $y'' \neq 0$.



Problem 2e



Problem 2f

f. $y = \dfrac{4x^2}{x+3}$

Domain all $x \neq -3$
x and y-intercept: $(0, 0)$.
Vertical asymptote: $x = -3$
Derivative: By the quotient rule, $y'(x) = \frac{4(2x(x+3)-x^2)}{(x+3)^2} = \frac{4x(x+6)}{(x+3)^2}$.
Critical points satisfy $y'(x) = 0$, so $x = 0$ and $x = -6$. When $x = 0$, $y = 0$ and is a minimum. When $x = -6$, $y = -48$ and is a maximum.
Second derivative $y''(x) = \frac{4((x^2+6x+9)(2x+6)-(x^2+6x)(2x+6))}{(x+3)^4} = \frac{36(2x+6)}{(x+3)^4}$. There is no point of inflection, as $y''(x) = 0$ at $x = -3$, the vertical asymptote.

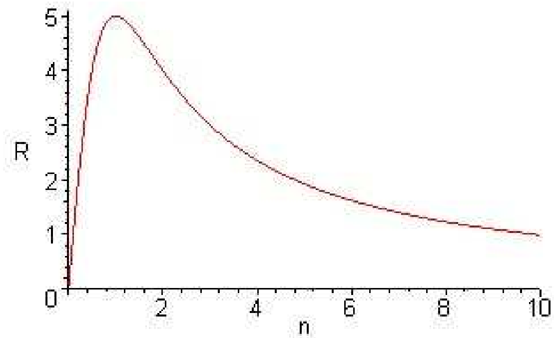
3. a. The rate of change in growth rate as a function of nutrient application is

$$\frac{dR}{dn} = -10\,\frac{n^2-1}{(1+n^2)^2}.$$

The rate of change in the growth rate when $n = 2$ is $R'(2) = -1.2$ (mm/day/mg/l)

b. There is a maximum at $n = 1$ (and a minimum occurs at $n = -1$, which is outside the domain) with $R(1) = 5$.

c. The n and R-intercept is $(0, 0)$, and there is a horizontal asymptote at $R = 0$. Below is a graph of the function.



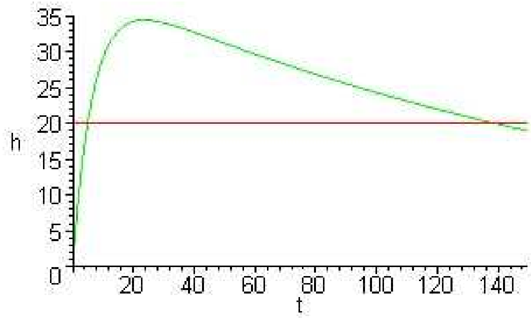
4. a. Since $h(t) = 40\left(e^{-0.005t} - e^{-0.15t}\right)$, it follows that

$$h'(t) = 40\left(-0.005e^{-0.005t} - (-0.15)e^{-0.15t}\right) = 40(0.15e^{-0.15t} - 0.005e^{-0.005t}).$$

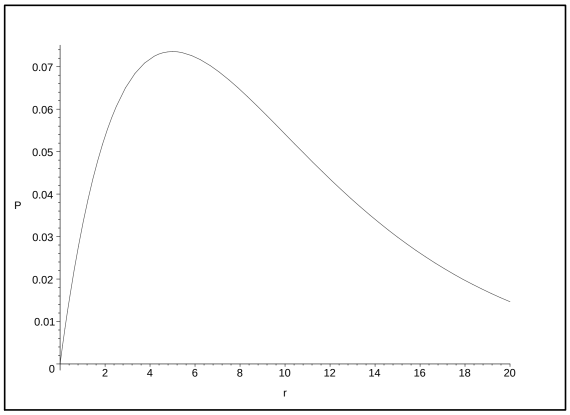
The maximum occurs when $h'(t) = 0$, so $0.15e^{-0.15t} = 0.005e^{-0.005t}$ or $e^{-0.005t}e^{0.15t} = \frac{0.15}{0.005}$. Thus, $e^{(.15-.005)t} = e^{0.145t} = 30$ or $0.145t = \ln(30)$. The maximum is at $t_c = \frac{1}{0.145}\ln(30) \simeq 23.46$ days. The maximum concentration is $h(t_c) = 40\left(e^{-0.005t_c} - e^{-0.15t_c}\right) = 34.39$ ng/dl.

b. The only intercept is $(0, 0)$. There is a horizontal asymptote at $h = 0$, since $\lim_{t\to\infty} h(t) = 0$. Maple can be used to show the $h(t) = 20$ at $t = 5.0$ and 138.6, so the hormone level remains above 20 ng/dl of blood for about 134 days. The graph is shown below.

5. By the product rule, the derivative is $P'(r) = 0.04e^{-0.2r} - 0.008re^{-0.2r}$.The maximum probability occurs when the derivative is zero, $0.04e^{-0.2r} - 0.008re^{-0.2r} = 0.04e^{-0.2r}(1-0.2r)$ or $0.2r = 1$. Thus, the maximum probability of a seed landing occurs at $r = 5$ m with a probability of $P(5) = 0.0736$.



The graph of the probability density function has an intercept at $(0,0)$ ($P(0) = 0$), a horizontal asymptote of $P = 0$ (since for large r, P becomes arbitrarily small), and a local maximum of $(5, 0.0736)$.



6. a. The equilibrium satisfies $N_e(0.8 - 0.04\ln(N_e)) = 0$. Since $N = 0$ is not in the domain. Thus, the equilibrium satisfies $0.04\ln(N_e) = 0.8$ or $\ln(N_e) = 20$. It follows that the equilibrium is $N_e = 4.852 \times 10^8$.

b. By the product rule, the derivative is $G'(N) = -N(0.04/N) + (0.8 - 0.04\ln(N)) = 0.76 - 0.04\ln(N)$. The maximum growth rate satisfies $0.76 - 0.04\ln(N) = 0$ or $\ln(N) = 19$. Thus, the maximum rate of growth occurs at $N_{max} = e^{19} = 1.785 \times 10^8$ with a maximum growth rate of $G(N_{max}) = 7.139 \times 10^6$.

c. Evaluating $G(2 \times 10^8) = 7.089 \times 10^6$, so the tumor is growing with this population of cells. Evaluating $G'(2 \times 10^8) = -0.004553$, so the rate of growth of the tumor is decreasing with this population of cells.

7. a. The concentration of glucose is given by $g(t) = 70 + 90e^{-0.7t}$, so for it to reach 90 mg/100 ml of blood, we need $90 = 70 + 90e^{-0.7t}$ or $20 = 90e^(- 0.7t)$. It follows that $e^{0.7t} = \frac{90}{20}$ or $0.7t = \ln\left(\frac{90}{20}\right)$. this takes about $t = 2.15$ hours. Note that this function has a g-intercept of 160 and a horizontal asymptote of $g = 70$. The graph for the concentration of glucose in the blood is below.

b. The rate of change of glucose per hour is

$$\frac{dg}{dt} = 0 + 90(-0.7)e^{-0.7t} = -63e^{-0.7t}.$$

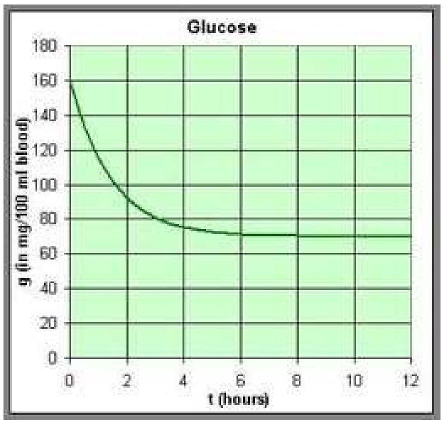
At $t = 1$, $g'(1) = -63e^{-0.7} = -31.28$ mg/100 ml of blood/hour.

c. The level of insulin satisfies the function $i(t) = 10(e^{-0.4t} - e^{-0.5t})$, so

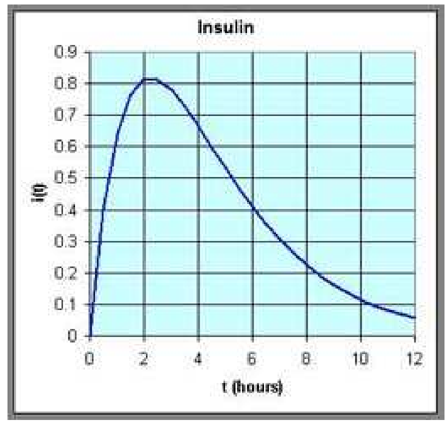
$$i'(t) = 10(-0.4e^{-0.4t} + 0.5e^{-0.5t}) = 5e^{-0.5t} - 4e^{-0.4t}.$$

The concentration is maximum where $i'(t) = 0$, so $5e^{-0.5t} = 4e^{-0.4t}$ or $\frac{5}{4} = e^{-0.4t}e^{0.5t} = e^{0.1t}$. It follows that $t = 10\ln\left(\frac{5}{4}\right) = 2.23$ hr. The maximum concentration is $i(2.23) = 10(e^{-0.4(2.23)} - e^{-0.5(2.23)}) = 0.819$. This graph starts at $(0, 0)$ and asymptotically approaches zero for large time. A graph of the insulin concentration is below also.

d. The rate of change of insulin per hour was computed above ($i'(t)$). The rate of change at $t = 1$ is $i'(1) = 5e^{-0.5} - 4e^{-0.4} = 0.351$ units/hour.



glucose



insulin

8. The radioactive decay of white lead (^{210}Pb) satisfies the equation $R(t) = R_0e^{-kt}$. With a half-life of 22 years, we have $R_0/2 = R_0e^{-22k}$, so $e^{22k} = 2$ or $22k = \ln(2)$. Thus, the decay constant $k = \frac{\ln(2)}{22} = 0.03151$ yr^{-1}. If the painting has 5% of the original amount of ^{210}Pb left, then $.05R_0 = R_0e^{-0.03151t}$ so $t = \frac{\ln(0.05)}{0.03151} = 95.1$ years old.

9. a. The colony of *Escherichia coli* satisfies $P(t) = 1000e^{0.01t}$, so to find doubling time we solve $1000e^{0.01t} = 2000$ or $e^{0.01t} = 2$. Thus, the doubling time is $0.01t = \ln(2)$ or $t = 100\ln(2) = 69.3$ min.

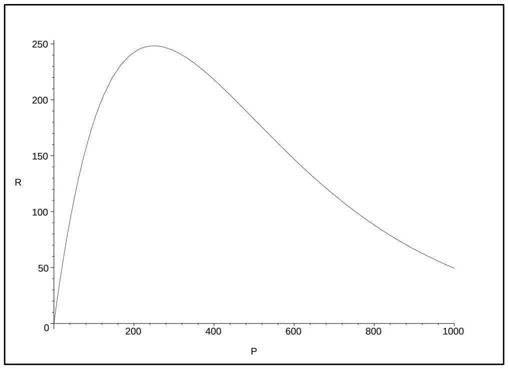
b. The mutant satisfies $M(t) = e^{kt}$ and doubles in 25 min. It follows that $e^{25k} = 2$ or $k = \frac{\ln(2)}{25} = 0.02773$. If the mutant colony is 20% of the population of the colony, then the original population is 4 times the mutant population. (20% mutant and 80% original). Thus, we must solve $1000e^{0.01t} = 4e^{kt}$ or $e^{kt}e^{-0.01t} = e^{(0.02773-0.01)t} = e^{0.01773t} = 250$. Thus, $0.01773t = \ln(250)$ or $t = \frac{\ln(250)}{k-0.01} = 311.5$ min.

c. The original population at $t = 500$ is $P(500) = 1000e^5 = 148,413$ bacteria. The mutant population is $M(500) = e^{500k} = 1,048,576$ bacteria. The rate of growth of the original population is $\frac{dP}{dt} = 1000(0.01)e^{0.01t} = 10e^{0.01t}$, which at $t = 500$ gives $P'(500) = 10e^5 = 1,484$ bacteria/min. The rate of growth of the mutant colony is $\frac{dM}{dt} = ke^{kt} = 0.02773e^{0.02773t}$, which at $t = 500$ gives $M'(500) = ke^{500k} = 0.02772e^13.86 = 29,073$ bacteria/min.

10. a. The derivative of the Ricker's updating function is

$$\frac{dR}{dP} = 2.7\,e^{-0.004P}(1 - 0.004\,P).$$

The R and P-intercept is the origin, $(0,0)$, and there is a horizontal asymptote at $R = 0$. There is a relative maximum at $P = 250$ with $R(P) = 675\,e^{-1} = 248.32$. Below is the graph.



b. The equilibria satisfy $P_e = 2.7P_e e^{-0.004P_e}$, so either $P_e = 0$ or $1 = 2.7e^{-0.004P_e}$. The latter gives $e^{0.004P_e} = 2.7$ or $P_e = 250\ln(2.7) = 248.31$.

c. If $P_e = 0$, then $R'(0) = 2.7 > 1$, so the equilibrium at $P_e = 0$ is unstable. If $P_e = 248.31$, then $R'(248.31) = 0.006748 < 1$, so the equilibrium at $P_e = 248.31$ is stable.

11. a. The derivative of the Beverton-Holt's updating function is

$$\frac{dB}{dP} = \frac{4(1 + 0.002P) - 4P(0.002)}{(1 + 0.002P)^2} = \frac{4}{(1 + 0.002P)^2}.$$

The B and P-intercept is the origin, $(0,0)$, and there is a horizontal asymptote at $B = 2000$. There are no extrema as this function is strictly increasing to its horizontal asymptote. Below is the graph.

b. The equilibria satisfy

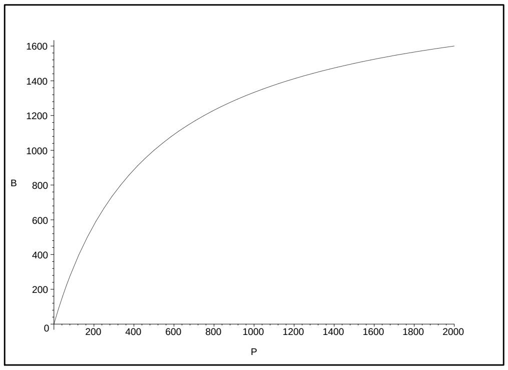
$$P_e = \frac{4P_e}{1 + 0.002P_e},$$

so either $P_e = 0$ or $1 + 0.002P_e = 4$. The latter gives $P_e = 1500$.

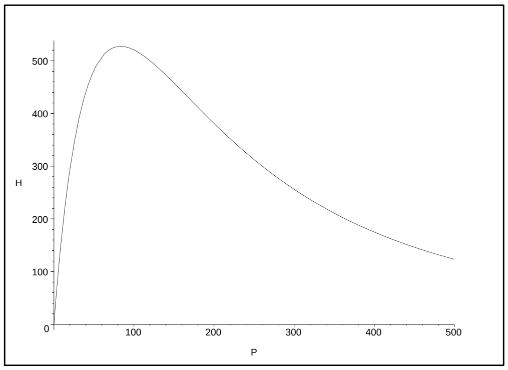
c. If $P_e = 0$, then $B'(0) = 4 > 1$, so the equilibrium at $P_e = 0$ is unstable. If $P_e = 1500$, then $B'(1500) = 0.25 < 1$, so the equilibrium at $P_e = 1500$ is stable.

12. a. The derivative of Hassell's updating function is

$$\frac{dH}{dP} = 20\frac{(1 + 0.004P)^4 \cdot 1 - 4P(1 + 0.004P)^3(0.004)}{(1 + 0.004P)^8} = \frac{20(1 - 0.012P)}{(1 + 0.004P)^5}.$$



The H and P-intercept is the origin, $(0,0)$, and there is a horizontal asymptote at $H = 0$. There is a critical point at $P_c = \frac{1}{0.012} = 83.333$ with $H(P_c) = 527.34$. This is clearly a maximum. Since the power of the denominator exceeds that of the numerator, there is a horizontal asymptote of $H = 0$. Below is the graph.



b. The equilibria satisfy

$$P_e = \frac{20P_e}{(1 + 0.004P_e)^4},$$

so either $P_e = 0$ or $(1 + 0.004P_e)^4 = 20$. The latter gives $P_e = 278.7$.

c. If $P_e = 0$, then $H'(0) = 20 > 1$, so the equilibrium at $P_e = 0$ is unstable. If $P_e = 278.7$, then $H'(278.7) = -1.1085 < -1$, so the equilibrium at $P_e = 278.7$ is unstable and oscillatory.

13. a. The equilibria satisfy

$$P_e = \frac{2P_e}{1 + 0.0025P_e^2} \quad \text{or} \quad P_e(1 + 0.0025P_e^2) = 2P_e.$$

Thus, either $P_e = 0$ or $1 + 0.0025P_e^2 = 2$. The latter implies that $0.0025P_e^2 = 1$ or $P_e^2 = 400$. Thus, $P_e = \pm 20$, but since the population density cannot be negative $P_e = 20$.

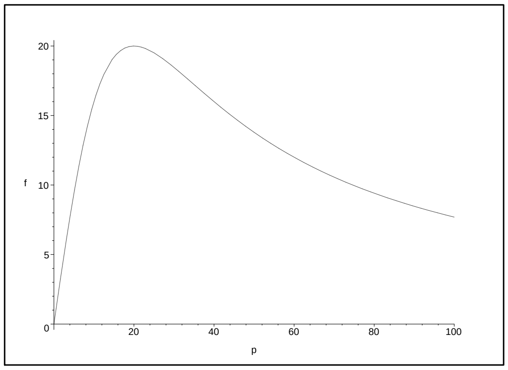
b. From the quotient rule,

$$f'(P_n) \quad = \quad \frac{(1 + 0.0025P_n^2)2 - 2P_n(0.005P_n)}{\left(1 + 0.0025\,{P_n}^2\right)^2}$$

$$= \frac{2 - 0.005\,{P_n}^2}{\left(1 + 0.0025\,{P_n}^2\right)^2}.$$

The maximum occurs when $f'(P_n) = 0$, which is when the numerator above is zero. Thus, $2 - 0.005P_n^2 = 0$ or $P_n^2 = 400$. It follows that the maximum mitotic increase occurs at $P_n = 20$, which is also the equilibrium.

c. A sketch of $f(P)$ is below. The only intercept is $(0, 0)$. As $P_n \to \infty$, the denominator of $f(P_n)$ gets larger faster than the numerator (higher power of P_n), so $f(P_n) \to 0$, so there is a horizontal asymptote at $P_{n+1} = 0$. From Part b., the maximum occurs at $(20, 20)$.



14. a. For the population model with the Allee effect, $N_{n+1} = N_n + 0.1N_n\left(1 - \frac{1}{9}(N_n - 5)^2\right)$ with (population in thousands) $N_0 = 4$, the next two generations are

$$N_1 = 4 + 0.1(4)\left(1 - \frac{1}{9}(4-5)^2\right) = 4.356$$

$$N_2 = 4.356 + 0.1(4.356)\left(1 - \frac{1}{9}(4.356-5)^2\right) = 4.771$$

in thousands of birds.

b. $N_e = N_e + 0.1N_e\left(1 - \frac{1}{9}(N_e - 5)^2\right)$, so $0.1N_e\left(1 - \frac{1}{9}(N_e - 5)^2\right) = 0$. Thus, $N_e = 0$ or $(N_e - 5)^2 = 9$. It follows that the equilibria are $N_e = 0$, 2, and 8.

c. From the expanded model, $N_{n+1} = A(N_n) = \frac{37}{45}N_n + \frac{1}{9}N_n^2 - \frac{1}{90}N_n^3$, the derivative is $A'(N) = \frac{37}{45} + \frac{2}{9}N - \frac{1}{30}N^2$. At $N_e = 0$, $A'(0) = \frac{37}{45}$, so this equilibrium is a stable equilibrium with solutions monotonically approaching 0. At $N_e = 2$, $A'(2) = \frac{17}{15}$, so this equilibrium is an unstable equilibrium with solutions monotonically moving away from 2. At $N_e = 8$, $A'(8) = \frac{7}{15}$, so this equilibrium is a stable equilibrium with solutions monotonically approaching 8.

d. Biologically, these results imply that if the population is below 2 thousand, then it will go to extinction ($N_e = 0$). If the population is above 2 thousand, then the population of birds will grow to a carrying capacity of $N_e = 8$ thousand.

15. a. From P_3, we have $P_3 = 68.34 = 28.49(1+r)^3$, so $(1+r) = (68.34/28.49)^{1/3} = 1.33863$. Thus, $r = 0.33863$. Doubling time satisfies $2P_0 = P_0(1+r)^n$ or $n = \ln(2)/\ln(1+r) = 2.377$ decades or 23.77 years.

b. The model predicts the population in 2000 is $P_5 = 28.49(1.33863)^5 = 122.46$ million. The percent error is $100\frac{(122.46-99.93)}{99.93} = 22.55\%$.

c. From the logistic model, we obtain $P_1 = 39.32$ million and $P_2 = 52.79$ million.

d. To find equilibria, we solve $P_e = 1.48P_e - 0.0035P_e^2$, which gives $P_e = 0$ or $P_e = 137.14$ million. The derivative of the updating function is $F'(P) = 1.48 - 0.007P$, so $F'(137.14) = 0.52$. It follows that this equilibrium is stable with solutions monotonically approaching this carrying capacity equilibrium.

16. a. Since $Y(t) = 1000(1 + 19e^{-0.1t})^{-1}$, the chain rule gives

$$\begin{aligned} Y'(t) &= -1000(1 + 19e^{-0.1t})^{-2}(-1.9e^{-0.1t}) \\ &= \frac{1900e^{-0.1t}}{(1 + 19e^{-0.1t})^2}. \end{aligned}$$

The second derivative is

$$\begin{aligned} Y''(t) &= 1900\frac{(1 + 19e^{-0.1t})^2(-0.1e^{-0.1t}) - e^{-0.1t}2(1 + 19e^{-0.1t})(-1.9e^{-0.1t})}{(1 + 19e^{-0.1t})^4} \\ &= \frac{190e^{-0.1t}(19e^{-0.1t} - 1)}{(1 + 19e^{-0.1t})^3}. \end{aligned}$$

The second derivative is 0 when $19e^{-0.1t} - 1 = 0$ or $e^{0.1t} = 19$. $t = 10\ln(19) = 29.44$. Thus, there is a point of inflection at $(29.44, 500)$.

b. Only intercept is $(0, 50)$. As $t \to \infty$, $e^{-0.1t} \to 0$, so $Y(t) \to 1000$, which gives a horizontal asymptote of $Y = 1000$. A graph of $Y(t)$ is below to the left. Since the population starts at 50, it doubles when it reaches 100. Solving $Y(t) = \frac{1000}{1+19e^{-0.1t}} = 100$ gives $1 + 19e^{-0.1t} = 10$, so $e^{0.1t} = \frac{19}{9}$. Thus, this population doubles when $t = 10\ln\left(\frac{19}{9}\right) = 7.47$ hr.

c. $Y(t)$ is increasing most rapidly at the point of inflection, so $t = 29.44$ hr. Substituting this value into the derivative gives the population increasing at a rate of 25 yeast/cc/hr. The only intercept is $(0, 4.75)$. Since the numerator has a decaying exponential function, the horizontal asymptote is $Y' = 0$. A sketch of $Y'(t)$ is below to the right. The maximum for $Y'(t)$ is $(29.44, 25)$.

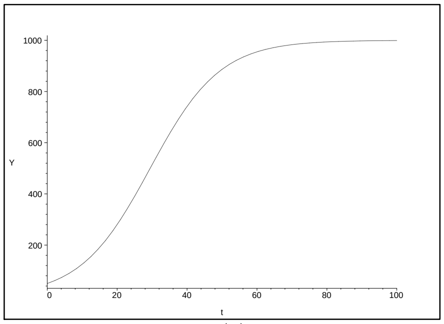
d. The Malthusian growth model doubles when it reaches 100. Solving $100 = 50e^{0.1t}$ gives $e^{0.1t} = 2$ or $t = 10\ln(2)$. Thus, the doubling time for the Malthusian growth model is $t = 6.93$ hr.

17. a. From the von Bertalanffy equation, it is easy to see that the graph passes through the origin, giving the t and L-intercepts to both be 0. As $t \to \infty$, $L(t) \to 16$, so there is a horizontal asymptote of $L = 16$. The graph of the length of the sculpin is below to the left.

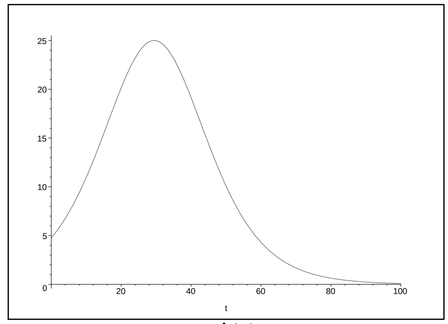
b. The composite function satisfies:

$$W(t) = 0.07\left(16(1 - e^{-0.4t})\right)^3 = 286.72(1 - e^{-0.4t})^3.$$

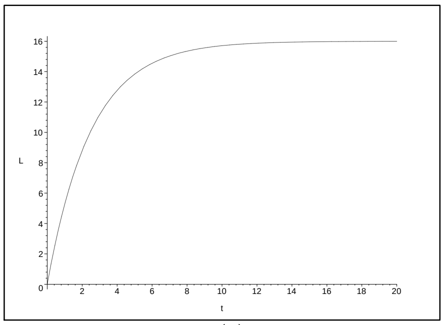
This function again passes through the origin, and it is easy to see that it has a horizontal asymptote at $W = 286.72$.



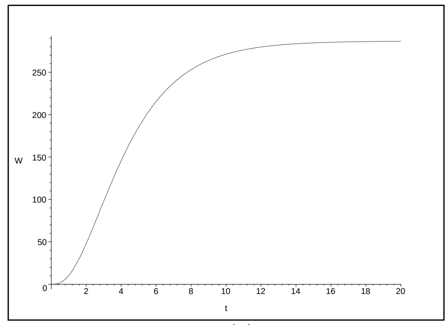
$Y(t)$



$Y'(t)$



$L(t)$



$W(t)$

c. We apply the chain rule to differentiate $W(t)$. The result is

$$W'(t) = 3 \cdot 286.72(1 - e^{-0.4t})^2(0.4)e^{-0.4t} = 344.064(1 - e^{-0.4t})^2 e^{-0.4t}.$$

The second derivative combines the product rule and the chain rule, giving:

$$\begin{aligned} W''(t) &= 344.064\left(-0.4(1 - e^{-0.4t})^2 e^{-0.4t} + 2(1 - e^{-0.4t})0.4e^{-0.4t}e^{-0.4t}\right) \\ &= 137.6256(1 - e^{-0.4t})e^{-0.4t}\left(-(1 - e^{-0.4t}) + 2e^{-0.4t}\right) \\ &= 137.6256(1 - e^{-0.4t})e^{-0.4t}\left(3e^{-0.4t} - 1\right). \end{aligned}$$

The point of inflection is when the sculpin has its maximum weight gain, and this occurs when

$$W''(t) = 137.6256(1 - e^{-0.4t})e^{-0.4t}\left(3e^{-0.4t} - 1\right) = 0.$$

or

$$(3e^{-0.4t} - 1) = 0 \quad \text{or} \quad e^{0.4t} = 3 \quad \text{or} \quad t = \frac{5\ln(3)}{2} \simeq 2.7465.$$

The maximum weight gain is

$$W'(2.7465) = 50.97 \text{ g/yr}.$$