1. (3pts) The radioactive decay equation satisfies:

$$\frac{dR}{dt} = -kR, \quad R(0) = 1, \qquad \text{so,} \qquad R(t) = e^{-kt},$$

where $k = \frac{\ln(2)}{5730} \approx 0.000121$ and $t$ is the age of the object. If $p$ is the fraction of $^{14}C$ remaining, then:

$$p = e^{-kt} \qquad \text{or} \qquad t = \frac{1}{k}\ln\left(\frac{1}{p}\right).$$

For the fossilized bone from a man in Western Pennsylvania, its age is approximately,

$$t = \frac{5730}{\ln(2)}\ln\left(\frac{1}{0.16}\right) \approx 15,150 \text{ yrs.}$$

With the error range of 15%–17%, its age ranges from 14,650 to 15,680 yrs, using the formula above with 0.15 or 0.17 replacing 0.16.
The formula above can be used to age the Kenyan man. Its age is approximately,

$$t = \frac{5730}{\ln(2)}\ln\left(\frac{1}{0.08}\right) \approx 20,880 \text{ yrs.}$$

With the error range of 7%–9%, its age ranges from 19,910 to 21,980 yrs, using the formula above with 0.07 or 0.09 replacing 0.08.

2. (6pts) This cascading system of radioactive elements is readily written as the following series of ODEs:

$$\begin{aligned} \dot{x}_1 &= r - k_1 x_1, & \text{with} \quad x_1(0) = 0, \\ \dot{x}_2 &= k_1 x_1 - k_2 x_2, & \text{with} \quad x_2(0) = 0, \\ \dot{x}_3 &= k_2 x_2 - k_3 x_3, & \text{with} \quad x_3(0) = 0, \end{aligned}$$

where

$$k_1 = \ln(2) \approx 0.69315, \qquad k_2 = \frac{\ln(2)}{10} \approx 0.069315, \qquad k_3 = \frac{\ln(2)}{400} \approx 0.0017329, \qquad r = 10.$$

These can readily be solved sequentially from our linear ODE techniques. The first equation has the integrating factor $\mu(t) = e^{k_1 t}$, so

$$\frac{d}{dt}\left(e^{k_1 t}x_1\right) = re^{k_1 t} \qquad \text{or} \qquad x_1(t) = e^{-k_1 t}\left(\frac{re^{k_1 t}}{k_1} + C_1\right).$$

With the initial condition, $x_1(0) = 0$, this becomes:

$$x_1(t) = \frac{r}{k_1}\left(1 - e^{-k_1 t}\right).$$

The second equation has the integrating factor $\mu(t) = e^{k_2 t}$, so

$$\frac{d}{dt}\left(e^{k_2 t}x_2\right) = re^{k_2 t}\left(1 - e^{-k_1 t}\right) \qquad \text{or} \qquad x_2(t) = e^{-k_2 t}\left(\frac{re^{k_2 t}}{k_2} - \frac{re^{(k_2 - k_1)t}}{k_2 - k_1} + C_2\right).$$

With the initial condition, $x_2(0) = 0$, this becomes:

$$x_2(t) = \frac{r}{k_2(k_2 - k_1)}\left(k_2 - k_1 - k_2 e^{-k_1 t} + k_1 e^{-k_2 t}\right).$$

The third equation has the integrating factor $\mu(t) = e^{k_3 t}$, so

$$\frac{d}{dt}\left(e^{k_3 t} x_3\right) = \frac{r e^{k_3 t}}{(k_2 - k_1)}\left(k_2 - k_1 - k_2 e^{-k_1 t} + k_1 e^{-k_2 t}\right)$$

or

$$x_3(t) = r e^{-k_3 t}\left(\frac{e^{k_3 t}}{k_3} - \frac{k_2 e^{(k_3 - k_1)t}}{(k_2 - k_1)(k_3 - k_1)} + \frac{k_1 e^{(k_3 - k_2)t}}{(k_2 - k_1)(k_3 - k_2)} + C_3\right).$$

With the initial condition, $x_3(0) = 0$, $C_3 = \frac{k_2}{(k_2 - k_1)(k_3 - k_1)} - \frac{1}{k_3} - \frac{k_1}{(k_2 - k_1)(k_3 - k_2)}$, so

$$x_3(t) = r\left(\frac{(1 - e^{-k_3 t})}{k_3} - \frac{k_2(e^{-k_1 t} - e^{-k_3 t})}{(k_2 - k_1)(k_3 - k_1)} + \frac{k_1(e^{-k_2 t} - e^{-k_3 t})}{(k_2 - k_1)(k_3 - k_2)}\right).$$

With the values of $r$, $k_1$, $k_2$, and $k_3$ substituted into the expressions for $x_1(t)$, $x_2(t)$, and $x_3(t)$, and evaluating these expressions at $t = 100$ and 400, we find that:

$$\begin{aligned} x_1(100) &= 14.427, & x_1(400) &= 14.427, \\ x_2(100) &= 144.11, & x_2(400) &= 144.27, \\ x_3(100) &= 781.41, & x_3(400) &= 2804.0. \end{aligned}$$

It is easy to see that these solutions require a fair amount of work!

3. (4pts) The *quasi-steady state* approximation for $x_1$ and $x_2$ assumes that $\dot{x}_1 = \dot{x}_2 = 0$, so $k_1\bar{x}_1 = r$ or $\bar{x}_1 = \frac{r}{k_1}$. Also, $k_2\bar{x}_2 = k_1\bar{x}_1 = r$ or $\bar{x}_2 = \frac{r}{k_2}$. This leaves the slow equation:

$$\dot{x}_3 \approx k_2\bar{x}_2 - k_3 x_3 = r - k_3 x_3 \qquad \text{with} \quad x_3(0) = 0.$$

This has the approximate solution:

$$x_3(t) \approx \frac{r}{k_3}\left(1 - e^{-k_3 t}\right),$$

which was significantly easier to solve. With this approximation, we have

$$x_3(100) \approx 918.15 \qquad \text{and} \qquad x_3(400) \approx 2885.39,$$

which produce 17% and 2.9% errors, respectively (with less than 0.6% error at $t = 1000$). The algebraic equations give $\bar{x}_1 = 14.427$ and $\bar{x}_2 = 144.27$. For larger times, these are good approximations for the system of equations.

4. (3pts) Consider the linear ODE given by:

$$t\frac{dy}{dt} - y = 3t^2\sin(t) \qquad \text{or} \qquad \frac{dy}{dt} - \frac{1}{t}y = 3t\sin(t).$$

The integrating factor for the equation is given by:

$$\mu(t) = e^{\int(-1/t)dt} = e^{-\ln(t)} = \frac{1}{t}.$$

It follows that

$$\frac{d}{dt}\left(\frac{y}{t}\right) = 3\sin(t).$$

Integrating both sides above gives:

$$\frac{y(t)}{t} = -3\cos(t) + C \qquad \text{or} \qquad y(t) = -3t\cos(t) + Ct.$$