Homework 4 – Method of Frobenius Due Mon. 11/8

Your HW covers the **Method of Frobenius** and must be turned in by **Mon. Nov 8 at Noon** in **Gradescope**.

Below lists the problems to be entered into **Gradescope** with the answers specifically requested. Your written answers must show the steps you take to find your power series solutions, including your recurrence relations. Answers simply found from Maple are not acceptable (though using it to check may help).

1. (8pts) For the initial-value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(0) = y_0, \quad y'(0) = v_0,$$

find a transformation $y = vY$ such that Y solves

$$Y'' + Q(x)Y = 0.$$

What are $v(x)$ and $Q(x)$? What do the initial conditions become for the $Y(x)$ version of the problem?

2. For the following problems, show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions (showing at least **4** non-zero terms in each series solution).

a. (10pts) $2x^2y'' + xy' + x^2y = 0$.

b. (10pts) $x^2y'' + 3xy' + (1 + x)y = 0$.

c. (10pts) $x^2y'' + 4xy' + (2 + x)y = 0$.

3. Bessel's equation of order $\frac{1}{2}$ is given by:

$$x^2y'' + xy' + \left(x^2 - \tfrac{1}{4}\right)y = 0.$$

This equation is important in solving partial differential equations with spherical geometry.

a. (10pts) Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions. Determine expressions for the coefficients for these two solutions.

b. (7pts) Consider the change of variables, $y(x) = x^{-\frac{1}{2}}v(x)$. Show that this change of variables reduces the Bessel's equation above to a much simpler ODE in $v(x)$. Solve this problem in v and determine the relatively simple closed form solutions, y, for Bessel's equation of order $\frac{1}{2}$, $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$.

4. Consider the following ODE:

$$x^2y'' + 6xy' + \left(6 - x^2\right)y = 0. \tag{1}$$

a. (10pts) Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions. Determine expressions for the coefficients for these two solutions.

b. (7pts) Consider the change of variables, $y(x) = x^{\alpha}v(x)$. With this change of variables, find α that eliminates any term with v', reducing (1) to a much simpler ODE in $v(x)$. Solve this problem in v and determine the relatively simple closed form solutions, y, for (1). Connect this solution to your series solution.

5. Consider the following ODE:

$$xy'' + (1 + 2x)y' + (1 + x)y = 0. \tag{2}$$

a. (10pts) Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions. Determine expressions for the coefficients for these two solutions.

b. (6pts) In Part a, $y_1(x)$ should be easily recognizable as a basic function. In Lecture Activity 8 (Problem 3) we introduced the **Reduction of Order** method as a means to find a 2^{nd} linearly independent solution, $y_2(x)$, given we know $y_1(x)$. Use this method to find $y_2(x)$ for (2) and compare this solution to your series solution in Part a.

6. Consider the following ODE:

$$xy'' - (2 + x^2)y' + xy = 0. \tag{3}$$

a. (10pts) Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions.

b. (6pts) Verify that one solution to (3) is

$$y_1(x) = e^{\frac{x^2}{2}}.$$

Does this match one of the solutions that you found in Part a? Use the **Reduction of Order** method like Problem 5 to find a second linearly independent solution.