1. Solve the following initial value problems. Answers may be left in implicit form when too hard to solve for $y(t)$.

a. $\dfrac{dy}{dt} = (2 - 0.2t)y, \quad y(0) = 10,$

b. $\dfrac{dy}{dt} = 2 - \dfrac{4}{t}, \quad y(1) = 5,$

c. $\dfrac{dy}{dt} = \dfrac{3t^2}{2y}, \quad y(0) = 4,$

d. $\dfrac{dy}{dt} = 0.02y\left(1 - \dfrac{y}{40}\right), \quad y(0) = 10,$

e. $(3t + 4y)\dfrac{dy}{dt} = 6t - 3y, \quad y(0) = 4,$

f. $t\dfrac{dy}{dt} - 2y = 4t^3\sin(4t), \quad y(1) = 2,$

g. $\dfrac{dy}{dt} = \dfrac{2\,ty}{t^2+1}, \quad y(0) = 3,$

h. $\dfrac{dy}{dt} = e^{t-y}, \quad y(0) = 6,$

i. $(e^t - 2y)\dfrac{dy}{dt} = 2 - ye^t, \quad y(0) = 6,$

j. $\dfrac{dy}{dt} + y = y^3e^t, \quad y(0) = 1,$

2. a. White lead is a pigment found in oil paints and can be used to detect art forgeries. In the absence of radium-226, lead-210 undergoes standard radioactive decay,

$$\frac{dP}{dt} = -kP.$$

Suppose that a sample from a painting has 10 disintegrations per minute in 1970 and then shows 8.5 disintegrations per minute in 1975. Find the half-life of lead-210 and give the value of k.

b. When there are impurities caused by radium-226 (which has a very long half-life), the differential equation for radioactive decay is modified to

$$\frac{dP}{dt} = -kP + r,$$

where $r = 0.25$ is source input from the radium-226 and k is from Part a. Solve this differential equation and determine the limit of P (disintegrations per minute of lead-210) as $t \to \infty$.

3. You are attending a conference, and the talks are going past the coffee break time. You really need a cup of tea (not liking coffee) to keep awake for the next set of talks. The refreshments are in a room that has a constant temperature of 21°C, and you find that the hot water is only 85°C. Five minutes later, the hot water is only 81°C.

a. Assume that the container of water satisfies Newton's law of cooling. ($H' = -k(H - T_e)$, where T_e is the environmental temperature.) If it was placed out when the talks were supposed to end with boiling water (water at 100°C), then how many minutes beyond the scheduled time did the talks go? (Hint: If $H(t)$ is the temperature, then use $H(0) = 85$ and $H(5) = 81$ to find the cooling constant k in Newton's law of cooling, then find when $H(t) = 100$.)

b. If tea needs water that is at least 93°C to give you enough caffeine for the next set of talks, then how long after the scheduled end of the talks can you wait?

4. a. An initially clean lake ($c(0) = 0$) concentrates pollution from an incoming stream because of evaporative loss of water of 200 m^3/day. The well-mixed lake has a stream flowing in at a rate of $f_1 = 2200\ \text{m}^3/\text{day}$ with a pesticide concentration of of $Q = 10$ ppb. The lake maintains a constant volume of $V = 10^6\ \text{m}^3$ by having a stream leaving with a flow of $f_2 = 2000\ \text{m}^3/\text{day}$. Find the differential equation describing the concentration of pesticide in the lake. Solve this differential equation.

b. Determine how long until the lake has a concentration of 5 ppb of pesticide. Also, find the limiting concentration of pesticide. Sketch a graph of the solution.

5. It has been shown that the radial spread of a disease in an orchard satisfies the differential equation

$$\frac{dT}{dt} = k\sqrt{T},$$

where T is the number of diseased trees and t is in years. Suppose that initially there is a single diseased tree ($T(0) = 1$) and that 4 years later $T(4) = 25$. Solve this differential equation. Find the value of k, then determine how many trees are infected after 10 years.

6. a. An elderly patient is running a fever of 39°C one night when her caregiver leaves. At 7 AM the next morning, the patient is found to have died from the fever. Her body temperature is taken immediately (at 7 AM) and found to be 35°C. Two hours later (at 9 AM), her body temperature is found to be 33.5°C. The room that she was staying was maintained at a constant temperature of 25°C. Assume that her body is cooled according Newton's Law of cooling,

$$\frac{dH}{dt} = -k_a(H - T_e),$$

where $H(t)$ is the temperature of the body, T_e is the room temperature, and k_a is the Newton's law cooling constant. Let $t = 0$ correspond to 7 AM, so $H(0) = 35$. Solve this differential equation, and use the information at 9 AM ($t = 2$) to find the value of k_a. Estimate the time of death assuming that the patient's temperature was 39°C at the time of death (using normal time, hours and minutes).

b. Since heat is lost through the surface, assume that we use a different model for the cooling of the body based on loss of heat through the surface. This model satisfies the differential equation:

$$\frac{dH}{dt} = -k_b(H - T_e)^{2/3}.$$

Once again, solve this differential equation, and use the information at 9 AM ($t = 2$) to find the value of k_b. Estimate the time of death under the same assumption above with this model.

7. a. A colony of bacteria grows according to the Malthusian growth model

$$\frac{dB}{dt} = 0.01\,B, \qquad B(0) = 1000,$$

where t is in min. Solve this differential equation and determine how long it takes for this population to double.

b. Because it takes a short time to adjust to the new medium, a better model is given by

$$\frac{dB}{dt} = 0.01(1 - e^{-t})\,B, \qquad B(0) = 1000.$$

Solve this differential equation.

c. Compare the populations predicted at $t = 5$ and 60 min.

8. a. Consider the Malthusian growth model for a particular animal that has recently colonized some region

$$\frac{dP}{dt} = 0.2\,P, \qquad P(0) = 100,$$

where t is in years. Solve this differential equation and determine how long it takes for this population to double.

b. Because of habitat encroachment, this animal is losing its range for expansion. This results in a growth rate that is time dependent. Suppose that the population satisfies the modified Malthusian growth model

$$\frac{dP}{dt} = (0.2 - 0.02\,t)\,P, \qquad P(0) = 100.$$

Solve this differential equation.

c. Find the maximum of this population and what year this occurs. Also, determine when the population returns to 100. Sketch a graph for this population.

9. a. Consider the growth of a population of cells in a declining medium. If the population growth depends on the absorption of the medium through the cell surface and the medium is decaying exponentially, then a differential equation for this population is given by

$$\frac{dP}{dt} = 0.3e^{-0.01t}P^{2/3}, \qquad P(0) = 1000,$$

where the initial population is 1000 and t is in hours. Solve this differential equation.

b. Find how long it takes for this population to double. What happens to this population for very large time (*i.e.*, find any horizontal asymptotes)? Sketch a graph for this population.

10. a. An initially clean pond that contains 10,000 m^3 of water maintains a constant volume. The stream flows in at a rate of 200 m^3/day with 10 $\mu\text{g/m}^3$ of phosphate (from fertilizer) in the stream. The pond is well-mixed, and a stream flows out at the same rate. Write a differential equation that describes the concentration of phosphate $c(t)$ in the lake, then solve this equation.

b. Algae grows well on phosphate. The rate of growth of algae is proportional to the concentration of phosphate and the population of algae $A(t)$ to the 2/3 power,

$$\frac{dA}{dt} = 0.05\,c(t)A^{2/3}, \qquad A(0) = 1000.$$

Find the population of algae at any time t.

11. Studies of Lake Apopka in Florida show that the alligators there have been exposed to high levels of various estrogen-simulating pesticides, like DDT (or its breakdown DDE), dieldrin, and toxaphenes. Apparently, this exposure has resulted in dramatic decrease in the size of the male alligator penises, which in turn finally spurred our Congress into action. (They were unconcerned when it was shown to have adverse effects on female animal populations.) The levels of these

estrogenic pesticides in this lake far exceed the safe levels established by the EPA. The effect on penis development is clearly related to the amount of exposure of the embryo, which in turn reflects the amount in the body of the female alligators.

a. The growth (weight) of a female alligator can be approximated by the following differential equation:

$$\frac{dw}{dt} = 0.2(80 - w), \qquad w(0) = 0,$$

where w is the weight in kg and t is the time in years. Solve this differential equation. Use the results to determine how long it takes to produce a mature 40 kg alligator.

b. The pesticides accumulate (especially in the fatty tissues) as the alligator grows and is not removed. Assume that the intake of pesticides is proportional to the weight of the alligator, so satisfies the differential equation

$$\frac{dP}{dt} = kw(t), \qquad P(0) = 0,$$

with $k = 600$ (μg/kg-yr) and P being the μg of pesticides in an alligator at Lake Apopka. Solve this differential equation. Find the amount of pesticide in an alligator that is 5 years old.

c. The concentration, $c(t)$ (in μg/g or ppm), is found by computing

$$c(t) = \frac{P(t)}{1000w(t)}.$$

Find the concentration in a 5 year old alligator. (Note that officials get concerned when pesticide levels reach 0.1 ppm in an animal.)

12. a. A new pesticide is introduced to a particular region, where a stream becomes contaminated with the pesticide and flows into a 10^6 m^3 lake. The stream flows at a rate of 4000 m^3/day with a concentration of 15 ng/m^3. Assuming that the lake is well-mixed and maintains a constant volume, then find the differential equation describing the concentration of this new pesticide in the lake. Solve this differential equation and find the limiting concentration in the lake.

b. More realistically, the flow of the stream is seasonal and fluctuates over the year by about 40%. The lake still maintains an almost constant volume, but a better model for the concentration of the pesticide in the lake is given by:

$$\frac{dc}{dt} = -0.001(4 - \cos(0.0172t))(c - 15), \qquad \text{with} \qquad c(0) = 0.$$

Solve this differential equation.

13. a. The decay of a particular fruit with total mass, M, satisfies the following differential equation:

$$\frac{dM}{dt} = -k\,M^{3/4}, \qquad M(0) = 16 \text{ g},$$

where t is in days. It is found that after 10 days only 1 g remains of the fruit, so $M(10) = 1$. Solve this differential equation. Find the value of k. Determine when the fruit completely vanishes ($M(t_f) = 0$).

b. A special culture of bacteria is added to the decaying fruit, and it is found that the decaying fruit satisfies the differential equation:

$$\frac{dM}{dt} = -0.8\, e^{-0.02t}\, M^{3/4}, \qquad M(0) = 16 \text{ g},$$

Solve this differential equation. Find the length of time for this fruit to completely vanish.

14. a. A population study is conducted on a new colony of invasive insects. The study finds that initially there are 60 of the insects in 1 m^2, $P(0) = 60$. Two weeks later a survey finds 80 of the insects in 1 m^2, $P(2) = 80$. Assume that this insect population is growing according to a Malthusian growth law:

$$\frac{dP}{dt} = r\, P,$$

where t is in weeks. Solve this differential equation, find the growth constant r, and determine how long it takes for the total population to double.

b. It is found that a predator is adapting to the new invasive insect and is learning to control this pest. A survey after four weeks finds the population has only increased to 90, $P(4) = 90$. The result is a declining growth rate and a better model for the population is given by the differential equation:

$$\frac{dP}{dt} = (a - bt)P.$$

Solve this differential equation. Use the data at $t = 0$, 2, and 4 weeks to find the constants a and b. Determine the time for this population to reach its maximum and what the maximum population is predicted to be.

15. Radioactive elements are often the products of the decay of another radioactive element. A differential equation describing this situation is given by the following:

$$\frac{dR}{dt} = -0.05R + 0.2e^{-0.01t}, \qquad R(0) = 10.$$

Find the solution to the initial value problem above.

16. (Allee effect) Suppose that a population, $P(t)$ (in thousands), is given by the model

$$\frac{dP}{dt} = P\left(9 - 0.01(P - 70)^2\right).$$

Sketch a graph of the right hand side of the differential equation, then draw the phase portrait. Find any equilibria and determine their stability. Find the carrying capacity for this particular population. Determine the critical threshold number of animals required to avoid extinction.

17. a. Suppose that a population of fish, $F(t)$ (in thousands), satisfies the logistic growth model given by

$$\frac{dF}{dt} = 0.4\, F\left(1 - \frac{F}{200}\right),$$

where t is in years. Solve this differential equation with $F(0) = 50$.

b. Find all equilibria for this model. Sketch a graph of the right hand side of the model, then draw the phase portrait on the F-axis. What is the stability of each of the equilibria? Determine the carrying capacity for this population of fish.

c. Assume that fishing is allowed and that 15,000 fish are harvested annually. The model becomes

$$\frac{dF}{dt} = 0.4\,F\left(1 - \frac{F}{200}\right) - 15.$$

Find all equilibria for this model. Sketch a graph of the right hand side of the model, then draw the phase portrait on the F-axis. What is the stability of each of the equilibria? Determine the carrying capacity for this population of fish. What is the threshold number of fish needed to avoid extinction?