Homework 2 – ODEs and Scaling Due Fri. 9/20

Work the following problems.

1. Consider the differential equation

$$\dot{v} + \tilde{k}\cos(\omega t)v = -g.$$

a. Using scaling arguments, show that you can turn this equation into

$$\frac{d\tilde{v}}{d\tau} + \gamma\cos(\tau)\tilde{v} = -1.$$

b. Solve this equation with initial condition $\tilde{v}(0) = \tilde{v}_0$. Does your initial condition matter as $\tau \to \infty$? In your solution, you should get the integral

$$\int_0^\tau e^{\gamma\sin(x)}dx.$$

Show that you cannot have a 2π periodic solution to the problem because of this integral.

c. Suppose that γ is small, which we denote by $\gamma \ll 1$. This means that γ^2 is much smaller than γ, γ^3 is smaller yet still and so forth. Using a Taylor series expansion for the integrand, show that you can approximate the integral as

$$\begin{aligned}\int_0^\tau e^{\gamma\sin(x)}dx =& \tau - \gamma(\cos(\tau) - 1) + \frac{\gamma^2}{4}\left(\tau - \frac{\sin(2\tau)}{2}\right)\\ &+ \frac{\gamma^3}{6}\left(\frac{2}{3} - \cos(\tau) + \frac{\cos^3(\tau)}{3}\right) + \mathcal{O}(\gamma^4).\end{aligned}$$

The notation $\mathcal{O}(\gamma^4)$ means all terms as small or smaller than γ^4, so basically everything you are truncating by not carrying the Taylor series out to an infinite number of terms. Which terms in this expansion are causing the solution to behave in a non-periodic fashion?

d. With your Taylor series expansions in γ, give your approximate solution, $\tilde{v}(\tau)$, up to and including terms of γ^2, *i.e.*,

$$\tilde{v}(\tau) = f_0(\tau) + \gamma f_1(\tau) + \gamma^2 f_2(\tau) + \mathcal{O}(\gamma^3),$$

finding f_0, f_1, and f_2.

e. Let $\gamma = 0.1$ and $\tilde{v}(0) = 10$. Graph the "exact" solution, using an ODE solver (such as MatLab's ODE23) and your expansions including only terms up to order γ and γ^2 (**2** truncations).

2. Using the transformation $x = \cos(\theta)$, rewrite the diffential equation

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \gamma y = 0,$$

in terms of the new independent variable θ.

3. Suppose you need to solve the problem

$$\frac{dy}{dt} + \alpha \cos(\omega t) y = \beta \sin(\omega t), \;\; y(0) = y_0,$$

and that you know that $\omega/\alpha = 200$.

a. Rescale the problem so that you can write it as

$$\frac{d\tilde{y}}{d\tau} + \epsilon \cos(\tau)\tilde{y} = \epsilon \sin(\tau), \;\; \tilde{y}(0) = \tilde{y}_0,$$

What is ϵ and why?

b. Solve this problem up to $\mathcal{O}(\epsilon^2)$, *i.e.*, find the function $\tilde{y}_1(\tau)$ so that the solution $\tilde{y}(\tau)$ is written as

$$\tilde{y}(\tau) = \tilde{y}_0 + \epsilon \tilde{y}_1(\tau) + \mathcal{O}(\epsilon^2)$$

What happens to $\tilde{y}_1(\tau)$ as τ becomes large?