# Math 537 - Ordinary Differential Equations

## Lecture Notes – Power Series

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## Outline







## Picard Iteration 1

Consider the *non-autonomous system of linear homogeneous differential equations*:

$$\dot{\mathbf{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A(x)\mathbf{y},$$

which does not have an obvious solution.

The proof of the *Existence and Uniqueness Theorem* often uses the *Method of Successive Approximations* or *Picard Iteration*.

For the **ODE**

$$\dot{\mathbf{y}} = \mathbf{f}(x, \mathbf{y}) \qquad \text{with} \quad \mathbf{y}(x_0) = \mathbf{y}_0,$$

Define

$$\begin{aligned} \phi_0(x) &= \mathbf{y}_0, \\ \phi_{k+1}(x) &= \mathbf{y}_0 + \int_{x_0}^{x} f(s, \phi_k(s))ds. \end{aligned}$$

Assuming the appropriate *continuity* and *continuity of the partial derivatives*, this sequence of iterates can be shown to converge to the *unique solution* of the **ODE**.





## Picard Iteration 2

**Picard Iteration**: Apply Picard iteration to the *initial value problem*:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \text{with} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}.$$

Let

$$\phi_0(x) = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}.$$

and

$$\phi_1(x) = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + \int_0^x \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} ds = \begin{pmatrix} a_0 + a_1 x \\ a_1 + a_0 \frac{x^2}{2} \end{pmatrix}.$$

Then

$$\phi_2(x) = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + \int_0^x \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} a_0 + a_1 s \\ a_1 + a_0 \frac{s^2}{2} \end{pmatrix} ds = \begin{pmatrix} a_0 + a_1 x + a_0 \frac{x^3}{2\cdot 3} \\ a_1 + a_0 \frac{x^2}{2} + a_1 \frac{x^3}{3} \end{pmatrix}.$$

$$\phi_1(x) = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + \int_0^x \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix} \begin{pmatrix} a_0 + a_1 s + a_0 \frac{s^3}{2\cdot 3} \\ a_1 + a_0 \frac{s^2}{2} + a_1 \frac{s^3}{3} \end{pmatrix} ds = \begin{pmatrix} a_0 + a_1 x + a_0 \frac{x^3}{2\cdot 3} + a_1 \frac{x^4}{3\cdot 4} \\ a_1 + a_0 \frac{x^2}{2} + a_1 \frac{x^3}{3} + a_0 \frac{x^5}{2\cdot 3\cdot 5} \end{pmatrix}.$$

## Airy's Equation



**Airy's Equation** arises in optics, quantum mechanics, electromagnetics, and radiative transfer:

$$y'' - xy = 0$$

Assume a **power series solution** of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

From before,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n,$$

which is substituted into the **Airy's equation**

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$





## Airy's Equation



**Airy's Equation**: The series can be written

$$2\cdot 1 a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n,$$

so $a_2 = 0$

The **recurrence relation** satisfies

$$(n+2)(n+1)a_{n+2} = a_{n-1} \qquad \text{or} \qquad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)},$$

so $a_2 = a_5 = a_8 = ... = a_{3n+2} = 0$ with $n = 0, 1, ...$

For the sequence, $a_0$, $a_3$, $a_6$, ... with $n = 1, 4, ...$

$$a_3 = \frac{a_0}{2\cdot 3}, \quad a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6}, \quad a_9 = \frac{a_6}{8\cdot 9} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$$





## Airy's Equation



**Airy's Equation**: The general formula is

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)}, \qquad n \ge 4$$

For the sequence, $a_1$, $a_4$, $a_7$, ... with $n = 2, 5, ...$

$$a_4 = \frac{a_1}{3\cdot 4}, \quad a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7}, \quad a_{10} = \frac{a_7}{9\cdot 10} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}$$

The general formula is

$$a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)}, \qquad n \ge 4$$



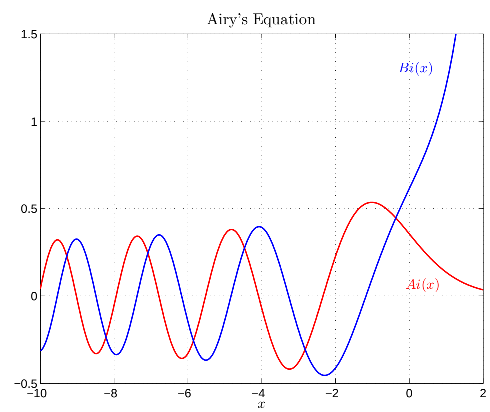


## Airy's Equation



**Airy's Equation**: The general solution is

$$\begin{aligned} y(x) &= a_0\left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots + \frac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)} + \cdots\right] \\ &\quad + a_1\left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots + \frac{x^{3n+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)} + \cdots\right] \end{aligned}$$



## Airy's Equation 5

**Airy's Equation**: The previous slide gave a power series solution, which we were able to obtain without trouble.

**Maple** gave us the solution:

$$y(x) = c_1 \mathrm{Ai}(x) + c_2 \mathrm{Bi}(x),$$

where the Airy's functions are defined by the *improper Riemann integrals*:

$$\begin{aligned} \mathrm{Ai}(x) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt \\ \mathrm{Bi}(x) &= \frac{1}{\pi}\int_0^\infty \left[\exp\left(\frac{-t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right] dt. \end{aligned}$$

However, these formula don't really give us any more information than the power series solutions about what is the geometric behavior of the solutions.

How do we obtain more information for much larger values of $x$?



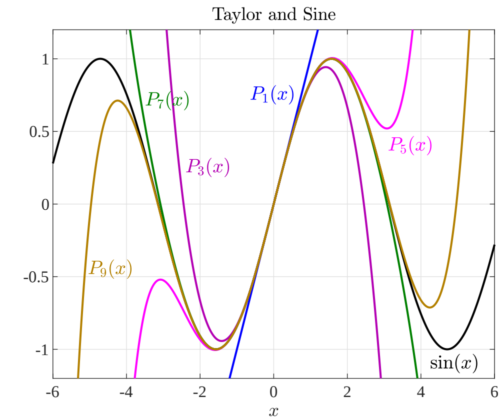


## Power Series and $\sin(x)$ 1

**Power Series**: In Calculus it was shown that

$$\sin(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j+1}.$$

The graph below shows $\sin(x)$ and truncated series to power $x^9$.








## Power Series and $\sin(x)$ 2

**Power Series**: Power series are relatively easy to produce, but they don't give us much about global behavior of the function.

Power series cannot show:

$$\sin(x + 2\pi) = \sin(x).$$

Power series cannot be used to show:

$$\sin\left(\tfrac{\pi}{2}\right) = 1.$$

Computing $\sin(x)$ for specific $x$ uses complex numerical algorithms for obtaining accurate values, which can include *Taylor's series*.





## Power Series and Airy's Function 1

The *power series* for ***Airy's equation*** gives us information near $x = 0$.

How do we learn more for $x \to \infty$?

Suppose we let $t = \frac{1}{x}$, then if $x \approx 0$, it follows that $t \to \infty$.

Similarly, if $t \approx 0$, it follows that $x \to \infty$.

Next consider the differential operators of ***Airy's equation*** with this change of variables:

$$\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = -\frac{1}{x^2}\frac{d}{dt} = -t^2\frac{d}{dt}$$

and

$$\frac{d^2}{dx^2} = \frac{2}{x^3}\frac{d}{dt} + \frac{1}{x^4}\frac{d^2}{dt^2} = 2t^3\frac{d}{dt} + t^4\frac{d^2}{dt^2}.$$

## Power Series and Airy's Function



Recall ***Airy's equation*** is given by

$$\frac{d^2y}{dx^2} - xy = 0.$$

***Airy's equation*** with the change of variables above becomes:

$$t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} - \frac{1}{t}y = 0.$$

Rewriting ***Airy's equation*** with this change of variables gives:

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} - \frac{1}{t^5}y = 0,$$

so for $t \approx 0$, these coefficients are unbounded, so this is a ***singular problem*** at $t = 0$.

This suggests problems for ***Airy's equation*** as $x \to \infty$.





## WKB Approximation



**WKB Approximation**: In Mathematical Physics when a ***linear differential equation*** has ***spatially varying coefficients***, then the wave-function $y$ is transformed into an ***exponential form***.

This semiclassical approximation is used in ***quantum mechanics*** and was developed by Gregor Wentzel, Hendrik Anthony Kramers, and Léon Brillouin in 1926 and Harold Jeffreys in 1923; hence, called the ***WKB approximation*** (or ***JWKB*** or ***WKBJ approximation***.

The transformation results in either the ***phase*** or ***amplitude*** becoming slow varying.

Specifically, the approximation causes the highest derivative to be multiplied by a small parameter, simplifying the analysis of the equation.





## WKB Approximation



**Airy's equation** is given by

$$\frac{d^2y}{dx^2} - xy = 0,$$

so let $y(x) = e^{s(x)}$ and this equation becomes:

$$\frac{d}{dx}\left(\frac{ds}{dx}e^s\right) - xe^s = 0,$$

or

$$\frac{d^2s}{dx^2}e^s + \left(\frac{ds}{dx}\right)^2 e^s - xe^s = 0.$$

Equivalently,

$$s'' + (s')^2 - x = 0.$$





## WKB Approximation



Now suppose that $s'' \ll (s')^2$, then the equation $s'' + (s')^2 - x = 0$ is approximated by:

$$(s')^2 - x \approx 0 \qquad \text{or} \qquad s' \approx \pm\sqrt{x}.$$

Integrating we have

$$s(x) \approx C \pm \tfrac{2}{3}x^{\frac{3}{2}}.$$

**Note**: If $s' \approx \pm\sqrt{x}$, then $s'' \approx \pm\frac{1}{2\sqrt{x}}$.

Thus, as $x \to \infty$, it is clear that $s'' \ll (s')^2$.

It follows that our initial guess of $y(x) = e^{s(x)}$ gives an approximate solution of

$$y(x) \approx c_1 e^{-\frac{2}{3}x^{\frac{3}{2}}} + c_2 e^{\frac{2}{3}x^{\frac{3}{2}}}, \qquad \text{as} \qquad x \to \infty.$$

## WKB Approximation



Similarly, as $x \to -\infty$,

$$s' \approx \pm i\sqrt{|x|}, \qquad \text{so} \qquad s(x) \approx C \pm \tfrac{2}{3} i |x|^{\frac{3}{2}},$$

which is still consistent with $s'' \ll (s')^2$.

Again it follows that our initial guess of $y(x) = e^{s(x)}$ gives an approximate solution of

$$y(x) \approx c_1 e^{-\frac{2}{3}i|x|^{\frac{3}{2}}} + c_2 e^{\frac{2}{3}i|x|^{\frac{3}{2}}}, \qquad \text{as} \qquad x \to -\infty.$$

By the Euler formula, this gives

$$y(x) \approx d_1 \cos\left(\tfrac{2}{3}|x|^{\frac{3}{2}}\right) + d_2 \sin\left(\tfrac{2}{3}|x|^{\frac{3}{2}}\right), \qquad \text{as} \qquad x \to -\infty.$$

It follows that for $x \to \infty$ the ***solution to Airy's equation*** grows or decays exponentially, while for $x \to -\infty$ it oscillates.



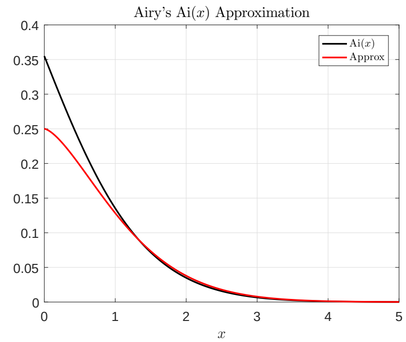


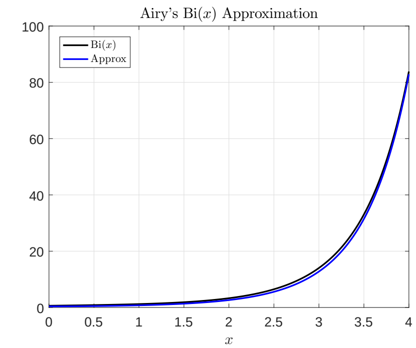
## WKB Approximation



Below we show graphs comparing our approximate solutions to the ***Airy's function*** with $x \geq 0$.

The left shows $y(x) = \text{Ai}(x)$ compared to $y_a(x) \approx 0.25\, e^{-\frac{2}{3}x^{\frac{3}{2}}}$, and the right shows $y(x) = \text{Bi}(x)$ compared to $y_b(x) \approx 0.4\, e^{\frac{2}{3}x^{\frac{3}{2}}}$.





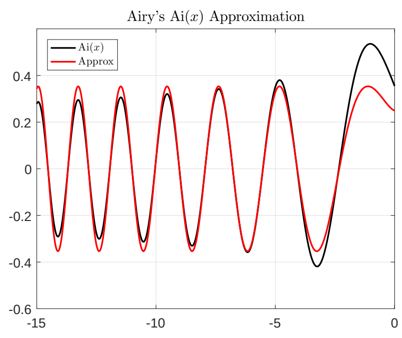


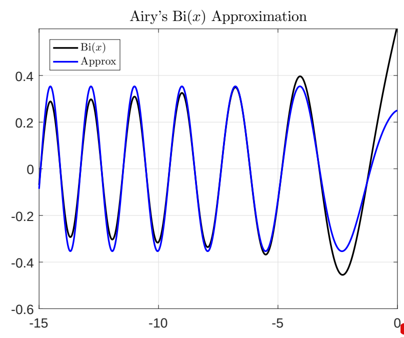

## WKB Approximation



Below we show graphs comparing our approximate solutions to the ***Airy's function*** with $x \leq 0$.

The left shows $y(x) = \text{Ai}(x)$ compared to $y_a(x) \approx 0.25\left(\cos\left(\tfrac{2}{3}|x|^{\frac{3}{2}}\right) + \sin\left(\tfrac{2}{3}|x|^{\frac{3}{2}}\right)\right)$, and the right shows $y(x) = \text{Bi}(x)$ compared to $y_b(x) \approx 0.25\left(\cos\left(\tfrac{2}{3}|x|^{\frac{3}{2}}\right) - \sin\left(\tfrac{2}{3}|x|^{\frac{3}{2}}\right)\right)$.








## WKB Approximation



The ***WKB Approximations*** are seen in the previous slides to match reasonably well away from $x = 0$ with the exponentials showing the appropriate growth or decay, while the oscillatory solutions match well in phase though the amplitude of the ***Airy's function*** are slowly decaying.

From the power series solutions, we have:

$$y_{1h}(x) = 1 + \frac{x^3}{2\cdot 3} + \dots \qquad \text{and} \qquad y_{2h}(x) = x + \frac{x^4}{3\cdot 4} + \dots,$$

so it follows that

$$\text{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt = c_1 y_{1h}(x) + c_2 y_{2h}(x)$$

for some constants $c_1$ and $c_2$.

Similarly,

$$\text{Bi}(x) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(\frac{-t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right] dt = d_1 y_{1h}(x) + d_2 y_{2h}(x)$$

for some constants $d_1$ and $d_2$.

But showing this is a non-trivial exercise in Complex Variables.

## Improved WKB Approximation



**Improved WKB Approximation**: Before we examined the equation:

$$s'' + (s')^2 - x = 0, \qquad (1)$$

with the assumption that $s'' \ll (s')^2$.

Now let $s = s_0 + s_1 + \dots$, where $s_0(x)$ satisfies $(s_0')^2 - x = 0$.

We take a two term expansion in Eq. (1) and obtain:

$$\begin{aligned} s_0'' + s_1'' + (s_0' + s_1')^2 - x &= 0, \\ s_0'' + s_1'' + (s_0')^2 + 2s_0's_1' + (s_1')^2 - x &= 0, \end{aligned}$$

which simplifies to

$$s_0'' + 2s_0's_1' + s_1'' + (s_1')^2 = 0.$$





## Improved WKB Approximation



**Improved WKB Approximation**: From before we had:

$$s_0'' + 2s_0's_1' + s_1'' + (s_1')^2 = 0.$$

Let us assume that $s_1''$ and $(s_1')^2$ are very small relative to the other two terms.

It follows that the next most dominant behavior after the ***WKB approximation*** for $s_0$ satisfies:

$$s_0'' + 2s_0's_1' \approx 0,$$

which is equivalent to

$$s_1' = -\frac{s_0''}{2s_0'} \qquad \text{or} \qquad s_1(x) = -\tfrac{1}{2}\ln(s_0').$$

Thus,

$$e^s = e^{s_0+s_1} \approx \frac{1}{(s_0')^{\frac{1}{2}}} e^{s_0}.$$





## Improved WKB Approximation

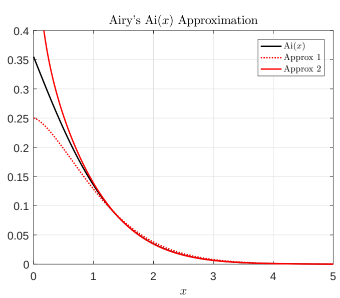


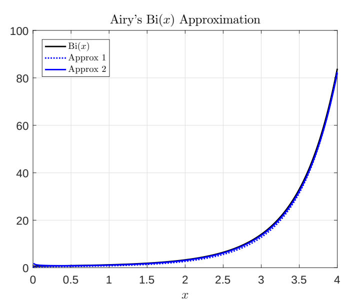
**Improved WKB Approximation**: From before we had $s_0'(x) = \pm x^{\frac{1}{2}}$, so it follows that

$$e^s \approx \frac{1}{(s_0')^{\frac{1}{2}}} e^{s_0} = \frac{1}{|x|^{\frac{1}{4}}} e^{\pm\frac{2}{3}x^{\frac{3}{2}}}.$$

This ***improved WKB approximation*** has little effect on the exponential growth or decay approximations of ***Airy's functions*** with $x \ge 0$.

The left shows $y(x) = \text{Ai}(x)$ compared to $y_a(x) \approx 0.27\, e^{-\frac{2}{3}x^{\frac{3}{2}}}/x^{\frac{1}{4}}$, and the right shows $y(x) = \text{Bi}(x)$ compared to $y_b(x) \approx 0.56\, e^{\frac{2}{3}x^{\frac{3}{2}}}/x^{\frac{1}{4}}$.





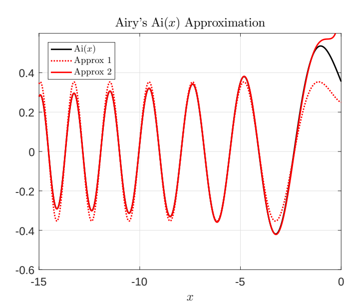


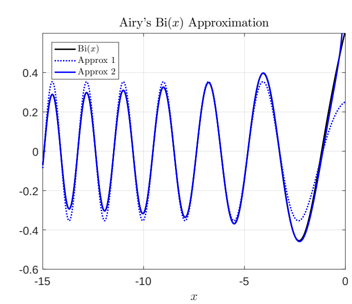

## Improved WKB Approximation



**Improved WKB Approximation**: However, the division by $|x|^{\frac{1}{4}}$ does affect amplitude, so this should be significant in the approximation for $x \le 0$.

The left shows $y(x) = \text{Ai}(x)$ compared to $y_a(x) \approx 0.4\left(\cos\left(\frac{2}{3}|x|^{\frac{3}{2}}\right) + \sin\left(\frac{2}{3}|x|^{\frac{3}{2}}\right)\right)/x^{\frac{1}{4}}$, and the right shows $y(x) = \text{Bi}(x)$ compared to $y_b(x) \approx 0.4\left(\cos\left(\frac{2}{3}|x|^{\frac{3}{2}}\right) - \sin\left(\frac{2}{3}|x|^{\frac{3}{2}}\right)\right)/x^{\frac{1}{4}}$.




Rapidly, we see the approximation almost identical to the ***Airy's functions***.

## Airy's Application



**Schrödinger Equation**: Consider the one-dimensional time-independent Schrödinger equation:

$$-\frac{\hbar}{2m}\frac{d^2y}{dx^2} + V(x)y = Ey,$$

where $y(x)$ gives standing wave solutions and $V(x)$ is the potential energy.

Rescaling readily transforms the ***Schrödinger equation*** into

$$-\frac{d^2y}{dx^2} + \tilde{V}(x)y = \tilde{E}y.$$

Suppose the potential energy satisfies $V(x) = x$, then equation becomes

$$\frac{d^2y}{dx^2} - (x - \tilde{E})y = 0.$$

With $\bar{x} = x - \tilde{E}$, (so $\frac{d}{dx} = \frac{d}{d\bar{x}}$)

$$\frac{d^2y}{d\bar{x}^2} - \bar{x}y = 0,$$

which has the solution:

$$y = c_1\,\mathrm{Ai}(\bar{x}) = c_1\,\mathrm{Ai}(x - \tilde{E}).$$





## Airy's Application



The solution to ***Schrödinger's equation*** was shown to be:

$$y = c_1\,\mathrm{Ai}(\bar{x}) = c_1\,\mathrm{Ai}(x - \tilde{E}),$$

where $\tilde{E}$ is the total energy.

This solution shows that crossing (right) the total energy threshold induces ***strong exponential decay***, which leads to ***quantum tunneling***.

The left side shows the classical solution with ***standing waves***.