Math 537 - Ordinary Differential Equations

Lecture Notes – Introduction to Math 537

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Outline

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Basic Information

Prerequisites: **Math 254** and **Math 337**

There is **NO TEXT** assigned to this course.

The topics are varied and come from multiple sources.

Lecture page on website will contain notes developed and a list of potential references and hyperlinks for this course.

Basic Information: Text/Topics

Course Topics

1. Linear Ordinary Differential Equations (ODEs) (Review)
2. Scaling ODEs
3. Fundamental Solutions (e^{At})
4. Power Series - Method of Frobenius
5. Singular Perturbation Methods
6. Multiple Time Scales (?)

Basic Information: Grading

Approximate Grading

Homework, including WeBWorK	36%
2 Midterms (Take-Home and In-class)	32%
Final	32%

- Homework includes electronic HW with WeBWorK and written problems (some inside WW problems). Critical to **keep up** on HW after each lecture.
- Exams are based heavily on HW problems and examples from lectures.
- Final: Monday, Dec 16, 8:00 – 10:00

Expectations and Procedures, I

- Most class attendance is OPTIONAL — Homework and announcements will be posted on the class web page. If/when you attend class:
  - Please be on time.
  - Please pay attention.
  - Please turn off cell/smart phones.
  - Please be courteous to other students and the instructor.
  - Abide by university statutes, and all applicable local, state, and federal laws.

Expectations and Procedures, II

- Please, turn in assignments on time. (The instructor reserves the right not to accept late assignments, and there is a maximum of **2** extensions of WeBWorK during the semester.)
- The instructor will make special arrangements for students with documented learning disabilities and will `try` to make accommodations for other unforeseen circumstances, *e.g.* illness, personal/family crises, etc. in a way that is fair to all students enrolled in the class. ***Please contact the instructor EARLY regarding special circumstances.***
- Students are expected ***and encouraged*** to ask questions in class!
- Students are expected ***and encouraged*** to to make use of office hours! If you cannot make it to the scheduled office hours: contact the instructor to schedule an appointment!

Expectations and Procedures, III

- Missed midterm exams: Don't miss exams! The instructor reserves the right to schedule make-up exams, modify the type and nature of this make-up, and/or base the grade solely on other work (including the final exam).
- Missed final exam: Don't miss the final! Contact the instructor ASAP or a grade of incomplete or F will be assigned.
- ***Academic honesty***: Submit your own work. Any cheating will be reported to University authorities and a **ZERO** will be given for that HW assignment or Exam.

MatLab

- Students can obtain **MatLab** from EDORAS Academic Computing.
- Google **SDSU MatLab** or access **https://edoras.sdsu.edu/ download/matlab.html**.
- **MatLab** and **Maple** can also be accessed in the **Computer Labs GMCS 421**, **422**, and **425** and the Library.
- A discounted student version of **Maple** is available (link available on the HW Assignment page).

Review of ODEs

Review from Math 337

- This course extends topics from Math 337: Elementary Ordinary Differential Equations (ODEs).
- Most problems relate to the ***linear ODEs***, scalar and systems.
- Differential equations are very important in many modeling situations, so often these connections will be pointed out.

Radioactive Decay

Radioactive Decay: Let $R(t)$ be the amount of a radioactive substance.

- Radioactive elements transition through decay into another state at a rate proportional to the amount of radioactive material present.
- It follows that the differential equation is:

$$\frac{dR(t)}{dt} = -k\,R(t) \quad \text{with} \quad R(0) = R_0.$$

- This has an exponential solution:

$$R(t) = R_0 e^{-kt}.$$

- Find k if the half-life of R is 8 da.
- **Solution:** $R(8) = \frac{R_0}{2} = R_0 e^{-8k}$, so

$$k = \frac{\ln(2)}{8} \approx 0.08664.$$

Mass-Spring 1

Harmonic Oscillator: A Hooke's law spring exerts a force that is proportional to the displacement of the spring.

- Newton's law of motion: Mass times the acceleration equals the force acting on the mass.
- The simplest spring-mass problem is

$$my'' = -ky \qquad \text{or} \qquad y'' + \omega^2 y = 0,$$

  where $\omega^2 = \frac{k}{m}$.
- This is a **second order**, **linear**, **homogeneous differential equation**.
- The ***characteristic equation*** is

$$\lambda^2 + \omega^2 = 0 \qquad \text{or} \qquad \lambda = \pm i\omega.$$

Mass-Spring 2

Mass-Spring: With ***eigenvalues***, $\lambda = \pm i\omega$ the general solution is:

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t),$$

where the constants c_1 and c_2 depend on the initial displacement and velocity.

We rewrite this 2^{nd} order ODE as a first order system by letting $y(t) = y_1(t)$ and $\dot{y}_1(t) = y_2(t)$, so

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -\omega^2 y_1. \end{aligned}$$

This becomes the 1^{st} order ODE system:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \text{or} \qquad \dot{\mathbf{y}} = A\mathbf{y}.$$

Mass-Spring 3

Mass-Spring: The ***characteristic equation*** and ***eigenvalues*** of A are the same as before:

$$\det|A - \lambda I| = \det \begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{vmatrix} = \lambda^2 + \omega^2 = 0,$$

so $\lambda = \pm i\omega$ (purely imaginary eigenvalues).

For $\lambda_1 = i\omega$, we have:

$$\begin{pmatrix} -\lambda_1 & 1 \\ -\omega^2 & -\lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -i\omega & 1 \\ -\omega^2 & -i\omega \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ i\omega \end{pmatrix}$.

We have $\lambda_2 = \bar{\lambda}_1$ and $\xi^{(2)} = \bar{\xi}^{(1)}$.

Mass-Spring 4

Mass-Spring: It follows that the vector solution is given by:

$$\begin{aligned} \mathbf{y}_1(t) &= \begin{pmatrix} 1 \\ i\omega \end{pmatrix} (\cos(\omega t) + i \sin(\omega t)) = \\ \mathbf{u}(t) + i\mathbf{w}(t) &= \begin{pmatrix} \cos(\omega t) \\ -\omega \sin(\omega t) \end{pmatrix} + i \begin{pmatrix} \sin(\omega t) \\ \omega \cos(\omega t) \end{pmatrix} \end{aligned}$$

Since the **real solution**, $\mathbf{u}(t)$, and the ***imaginary solution***, $\mathbf{w}(t)$, are linearly independent solution, we take the linear combination to obtain the general **real solution**:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(\omega t) \\ -\omega \sin(\omega t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\omega t) \\ \omega \cos(\omega t) \end{pmatrix}.$$

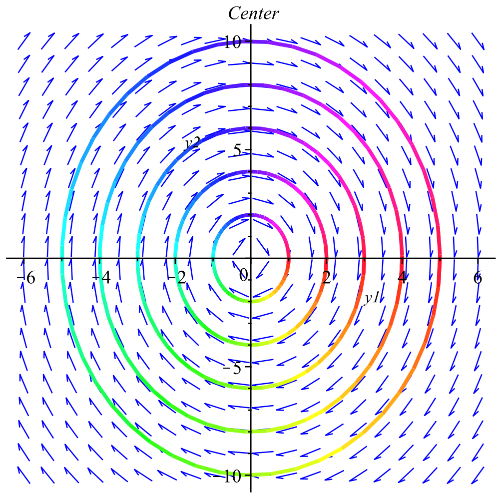
Mass-Spring 5

Mass-Spring: If we let $\omega = 2$, then the figure below gives the **phase portrait** for this system.

This is a **center**.

All solutions form ellipses around the origin.

The horizontal axis is the displacement, while the vertical axis is the velocity.



Chebyshev's Equation 1

Chebyshev's Equation is given by

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0$$

Let $\alpha = 4$ and try a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{so} \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

These are inserted into the **Chebyshev Equation** to give:

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + 16 \sum_{n=0}^{\infty} a_n x^n = 0$$

Note that the first two sums could start their index at $n = 0$ without changing anything

Chebyshev's Equation 2

Chebyshev's Equation: The previous expression is easily changed by multiplying by x or x^2 and shifting the index to:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + 16\sum_{n=0}^{\infty} a_n x^n = 0$$

Equivalently,

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n(n-1) + n - 16)\, a_n\right] x^n = 0$$

or

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - \left(n^2 - 16\right) a_n\right] x^n = 0$$

Chebyshev's Equation 3

Chebyshev's Equation: The previous expression gives the **recurrence relation**:

$$a_{n+2} = \frac{n^2 - 16}{(n+2)(n+1)} a_n \qquad \text{for} \quad n = 0, 1, ..$$

As before, a_0 and a_1 are arbitrary with $y(0) = a_0$ and $y'(0) = a_1$

It follows that

$$a_2 = -\frac{16}{2}a_0 = -8a_0, \qquad a_4 = \frac{4-16}{4\cdot 3}a_2 = 8a_0, \qquad a_6 = 0 = a_8 = ... = a_{2n}$$

and

$$a_3 = -\frac{15}{3\cdot 2}a_1 = -\frac{5}{2}a_1, \qquad a_5 = -\frac{7}{5\cdot 4}a_3 = \frac{7}{8}a_1, \qquad a_7 = \frac{9}{7\cdot 6}a_5 = \frac{3}{16}a_1, ...$$

Chebyshev's Equation

Chebyshev's Equation with $\alpha = 4$: From the **recurrence relation**, we see that the even series terminates after x^4, leaving a 4^{th} order polynomial solution.
The general solution becomes:

$$\begin{aligned} y(x) &= a_0\left(1 - 8x^2 + 8x^4\right) \\ &+ a_1\left(x - \frac{5}{2}x^3 + \frac{7}{8}x^5 + \frac{3}{16}x^7 + ...\right) \end{aligned}$$

$$\begin{aligned} y(x) &= a_0\left(1 - 8x^2 + 8x^4\right) \\ &+ a_1\left(x + \sum_{n=1}^{\infty}\frac{[(2n-1)^2-16][(2n-3)^2-16]\cdot\dots\cdot(3^2-16)(1-16)}{(2n+1)!}x^{2n+1}\right) \end{aligned}$$

Chebyshev's Equation

Chebyshev's Equation: More generally, it is not hard to see that for any α an integer, the **Chebyshev's Equation** results in one solution being a polynomial of order α (only odd or even terms). The other solution is an infinite series.

The polynomial solution converges for all x, while the infinite series solution converges for $|x| < 1$.

In this course we'll examine what happens when the ***power series*** is expanded around a ***singular point***.