Math 537 - Ordinary Differential Equations

Lecture Notes – Method of Frobenius

Joseph M. Mahaffy,
⟨jmahaffy@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

Fall 2019

Outline

Definitions 1

Consider the 2^{nd} ***order linear differential equation***:

$$P(x)y'' + Q(x)y' + R(x)y = F(x). \tag{1}$$

Definition (Ordinary and Singular Points)

x_0 is an ***ordinary point*** of Eqn. (1) if $P(x_0) \neq 0$ and $Q(x)/P(x)$, $R(x)/P(x)$, and $F(x)/P(x)$ are ***analytic*** at x_0.

x_0 is a ***singular point*** of Eqn. (1) if x_0 is not an ***ordinary point***.

The previous ***ODEs*** solved by ***power series*** methods have centered around $x_0 = 0$, when this is an ***ordinary point***.

In an interval about a ***singular point***, the solutions of Eqn. (1) can exhibit behavior different from ***power series solutions*** for Eqn. (1) near an ***ordinary point***.

If $x_0 = 0$, then these solutions may behave like $\ln(x)$ or x^{-n} near x_0.

Definitions 2

We concentrate on the ***homogeneous*** 2^{nd} ***order linear differential equation*** to better understand behavior of the solution near a ***singular point***:

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{2}$$

Definition (Regular and Irregular Singular Points)

If x_0 is a ***singular point*** of Eqn. (2), then x_0 is a ***regular singular point*** provided the functions:

$$(x - x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad (x - x_0)^2\frac{R(x)}{P(x)}$$

are ***analytic*** at x_0.

A ***singular point*** that is **not regular** is said to be an ***irregular singular point***.

If Eqn. (2) has a ***regular singular point*** at x_0, then it is **possible** that **no power series** solution exists of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Bessel's Equation

Example: Consider ***Bessel's equation of order ν***:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where $P(x) = x^2$, $Q(x) = x$, and $R(x) = x^2 - \nu^2$.

It is clear that $x = 0$ is a ***singular point***.

We see that

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = 1 \qquad \text{and} \qquad \lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0}(x^2 - \nu^2) = -\nu^2,$$

which are both finite, so ***analytic***.

It follows that $x_0 = 0$ is a ***regular singular point***.

Any other value of x_0 for ***Bessel's equation*** gives an ***ordinary point***.

Legendre's Equation 1

Example: Consider ***Legendre's equation***:

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where $P(x) = (1 - x^2)$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha + 1)$.

It is clear that $x = \pm 1$ are ***singular points***.

We see that

$$\lim_{x\to 1}(x-1)\frac{Q(x)}{P(x)} = \lim_{x\to 1}(x-1)\frac{-2x}{(1-x^2)} = \lim_{x\to 1}\frac{2x}{1+x} = 1, \qquad \text{and}$$

$$\lim_{x\to 1}(x-1)^2\frac{R(x)}{P(x)} = \lim_{x\to 1}(x-1)^2\frac{\alpha(\alpha+1)}{(1-x^2)} = \lim_{x\to 1}(x-1)\frac{-\alpha(\alpha+1)}{1+x} = 0,$$

which are both finite, so ***analytic***.

It follows that $x_0 = 1$ is a ***regular singular point***, and a similar argument shows that $x_0 = -1$ is a ***regular singular point***.

Legendre's Equation 2

Any other value of x_0 for ***Legendre's equation*** gives an ***ordinary point***, so $x_0 = 0$ is an ***ordinary point***, and we seek power series solutions:

$$y(x) = \sum_{n=0}^{\infty} a_nx^n, \qquad y'(x) = \sum_{n=1}^{\infty} na_nx^{n-1}, \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$$

These are inserted into the ***Legendre Equation*** to give:

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - 2x\sum_{n=1}^{\infty} na_nx^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_nx^n = 0$$

The first two sums could start their index at $n = 0$ without changing anything, so this expression is easily changed by multiplying by x or x^2 and shifting the index to:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_nx^n - 2\sum_{n=0}^{\infty} na_nx^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_nx^n = 0.$$

Legendre's Equation 3

Collecting coefficients gives:

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)a_{n+2} - \big(n(n-1) + 2n - \alpha(\alpha+1)\big)a_n\Big]x^n = 0$$

or

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)a_{n+2} - \big(n(n+1) - \alpha(\alpha+1)\big)a_n\Big]x^n = 0.$$

The previous expression gives the **recurrence relation**:

$$a_{n+2} = \frac{n(n+1) - \alpha(\alpha+1)}{(n+2)(n+1)}a_n \qquad \text{for} \quad n = 0, 1, ..$$

Properties of power series give a_0 and a_1 as arbitrary with $y(0) = a_0$ and $y'(0) = a_1$.

Legendre's Equation 4

The **recurrence relation** shows that all the even coefficients, a_{2n}, depend only on a_0, while all odd coefficients, a_{2n+1}, depend only on a_1, so all solutions have the form:

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where $y_1(x)$ has only even powers of x and $y_2(x)$ has only odd powers of x.

From the **recurrence relation** it is clear that any integer value of $\alpha = 0, 1, 2, \dots$ results in coefficients $a_{\alpha+2} = a_{\alpha+4} = \dots = a_{\alpha+2k} = 0$ for $k = 1, 2, \dots$

This results in one solution being an *α-degree polynomial*, which is valid for all x.

The other solution remains an *infinite series*.

If α is not an integer, then both *linearly independent solutions* are *infinite series*.

The *polynomial solution* converges for all x, while the *infinite series solution* converges for $|x| < 1$ using the *ratio test*.

Cauchy-Euler Equation 1

Cauchy-Euler Equation (Also, **Euler Equation**): Consider the differential equation:

$$L[y] = t^2 y'' + \alpha t y' + \beta y = 0,$$

where α and β are constants.

Assume $t > 0$ and attempt a solution of the form

$$y(t) = t^r.$$

Note that t^r may not be defined for $t < 0$.

The result is

$$\begin{aligned} L[t^r] &= t^2(r(r-1)t^{r-2}) + \alpha t(rt^{r-1}) + \beta t^r \\ &= t^r[r(r-1) + \alpha r + \beta] = 0. \end{aligned}$$

Thus, obtain quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0.$$

Cauchy-Euler Equation 2

Cauchy-Euler Equation: The quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0$$

has roots

$$r_1, r_2 = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}.$$

This is very similar to our **constant coefficient homogeneous** DE.

Real, Distinct Roots: If $F(r) = 0$ has real roots, r_1 and r_2, with $r_1 \neq r_2$, then the **general solution** of

$$L[y] = t^2 y'' + \alpha y' + \beta y = 0,$$

is

$$y(t) = c_1 t^{r_1} + c_2 t^{r_2}, \qquad t > 0.$$

Cauchy-Euler Equation 3

Example: Consider the equation

$$2t^2 y'' + 3ty' - y = 0.$$

By substituting $y(t) = t^r$, we have

$$t^r[2r(r-1) + 3r - 1] = t^r(2r^2 + r - 1) = t^r(2r-1)(r+1) = 0.$$

This has the real roots $r_1 = -1$ and $r_2 = \frac{1}{2}$, giving the **general solution**

$$y(t) = c_1 t^{-1} + c_2\sqrt{t}, \qquad t > 0.$$

Cauchy-Euler Equation

Equal Roots: If $F(r) = (r - r_1)^2 = 0$ has r_1 as a double root, there is one solution, $y_1(t) = t^{r_1}$.

Need a second linearly independent solution.

Note that not only $F(r_1) = 0$, but $F'(r_1) = 0$, so consider

$$\begin{aligned} \frac{\partial}{\partial r} L[t^r] &= \frac{\partial}{\partial r}[t^r F(r)] = \frac{\partial}{\partial r}[t^r (r - r_1)^2] \\ &= (r - r_1)^2 t^r \ln(t) + 2(r - r_1)t^r. \end{aligned}$$

Also,

$$\frac{\partial}{\partial r} L[t^r] = L\left[\frac{\partial}{\partial r}(t^r)\right] = L[t^r \ln(t)].$$

Evaluating these at $r = r_1$ gives

$$L[t^{r_1} \ln(t)] = 0.$$

Cauchy-Euler Equation

Equal Roots: For $F(r) = (r - r_1)^2 = 0$, where r_1 is a double root, then the differential equation

$$L[y] = t^2 y'' + \alpha y' + \beta y = 0,$$

was shown to satisfy

$$L[t^{r_1}] = 0 \qquad \text{and} \qquad L[t^{r_1} \ln(t)] = 0.$$

It follows that the **general solution** is

$$y(t) = (c_1 + c_2 \ln(t))t^{r_1}.$$

Cauchy-Euler Equation

Example: Consider the equation

$$t^2 y'' + 5ty' + 4y = 0.$$

By substituting $y(t) = t^r$, we have

$$t^r[r(r-1) + 5r + 4] = t^r(r^2 + 4r + 4) = t^r(r+2)^2 = 0.$$

This only has the real root $r_1 = -2$, which gives **general solution**

$$y(t) = (c_1 + c_2 \ln(t))t^{-2}, \qquad t > 0.$$

Cauchy-Euler Equation

Complex Roots: Assume $F(r) = 0$ has $r = \mu \pm i\nu$ as complex roots, the solutions are still $y(t) = t^r$.

However,

$$t^r = e^{(\mu + i\nu)\ln(t)} = t^\mu[\cos(\nu \ln(t)) + i \sin(\nu \ln(t))].$$

As before, we obtain the two linearly independent solutions by taking the real and imaginary parts, so the **general solution** is

$$y(t) = t^\mu[c_1 \cos(\nu \ln(t)) + c_2 \sin(\nu \ln(t))].$$

Cauchy-Euler Equation

Example: Consider the equation

$$t^2y'' + ty' + y = 0.$$

By substituting $y(t) = t^r$, we have

$$t^r[r(r-1) + r + 1] = t^r(r^2 + 1) = 0.$$

This has the complex roots $r = \pm i$ ($\mu = 0$ and $\nu = 1$), which gives the **general solution**

$$y(t) = c_1\cos(\ln(t)) + c_2\sin(\ln(t)), \qquad t > 0.$$

Regular Singular Problem

Regular Singular Point: Consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

and without loss of generality assume that it has a ***regular singular point*** at $x_0 = 0$.

This implies that $xQ(x)/P(x) = p(x)$ and $x^2R(x)/P(x) = q(x)$ are ***analytic*** at $x = 0$, so

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \text{and} \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

are convergent series for some interval $|x| < \rho$ with $\rho > 0$.

This gives the equation:

$$L[y] = x^2y'' + xp(x)y' + q(x)y = 0.$$

Regular Singular Problem

Regular Singular Problem: Since the $p(x)$ and $q(x)$ are ***analytic*** at $x = 0$, the ***second order linear equation*** can be written:

$$L[y] = x^2y'' + x\left(\sum_{n=0}^{\infty} p_n x^n\right)y' + \left(\sum_{n=0}^{\infty} q_n x^n\right)y = 0. \tag{3}$$

Note that if $p_n = q_n = 0$ for $n = 1, 2, \dots$ with

$$p_0 = \lim_{x\to 0}\frac{xQ(x)}{P(x)} \qquad \text{and} \qquad q_0 = \lim_{x\to 0}\frac{x^2R(x)}{P(x)},$$

then the ***second order linear equation*** becomes the ***Cauchy-Euler equation***:

$$x^2y'' + xp_0y' + q_0y = 0.$$

Method of Frobenius

Method of Frobenius: Since the **regular singular problem** starts with the ***zeroth order terms*** in the coefficients being similar to the ***Cauchy-Euler equation***, this suggests looking for solutions with terms of x^r.

As with the ***Cauchy-Euler equation***, we consider $x > 0$ with the case $x < 0$ handled by a change of variables $x = -\xi$ with $\xi > 0$.

The **Method of Frobenius** seeks solutions of the form:

$$y(x) = x^r(a_0 + a_1x + \cdots + a_nx^n + \dots) = x^r\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n}.$$

1. What values of r give a solution to (3) in the above form?
2. What is the ***recurrence relation*** for the a_n?
3. What is the radius of convergence for the above series?

Method of Frobenius 2

For the **regular singular problem**

$$x^2y'' + x\left(\sum_{n=0}^{\infty} p_n x^n\right)y' + \left(\sum_{n=0}^{\infty} q_n x^n\right)y = 0,$$

we seek a solutions of the form: $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$, so

$$y' = \sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}.$$

Thus,

$$\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n} + \left(\sum_{n=0}^{\infty} p_n x^n\right)\sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \left(\sum_{n=0}^{\infty} q_n x^n\right)\sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

Method of Frobenius 3

Indicial Equation: In the previous equation, we examine the lowest power of x, so $n = 0$.

This gives

$$a_0 x^r \big(r(r-1) + p_0 r + q_0\big) = 0.$$

For $a_0 \neq 0$, we obtain the ***indicial equation***, which came from solving the ***Cauchy-Euler equation***:

$$F(r) = r(r-1) + p_0 r + q_0 = 0,$$

which is a ***quadratic equation***.

The form of the solution of the ***Cauchy-Euler equation*** depended on the values of r for the ***indicial equation***, which in turn affects the factor x^r multiplying our ***power series solution***.

Distinct Roots, $r_1 - r_2 \neq N$ 1

The ***Method of Frobenius*** breaks into **3 cases**, depending on the roots of the ***indicial equation***.

Case 1. Distinct roots not differing by an integer, $r_1 - r_2 \neq N$.

For this case, a basis for the solution of the ***regular singular problem*** satisfies:

$$y_1(x) = x^{r_1}(a_0 + a_1x + \dots + a_nx^n + \dots) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$$

and

$$y_2(x) = x^{r_2}(b_0 + b_1x + \dots + b_nx^n + \dots) = x^{r_2}\sum_{n=0}^{\infty} b_n x^n$$

with these solutions converging for at least $|x| < \rho$, where ρ is the radius of convergence for $p(x)$ and $q(x)$.

Distinct Roots, $r_1 - r_2 \neq N$ 2

Example: Consider the ***regular singular problem*** given by

$$4xy'' + 2y' + y = 0,$$

where $x = 0$ is a ***regular singular point***.

From our definitions before we have $p(x) = \frac{1}{2}$ and $q(x) = \frac{x}{4}$, which implies that $p_0 = \frac{1}{2}$ and $q_0 = 0$.

Since p and q have convergent power series for all x, the solutions will converge for $|x| < \infty$.

The ***indicial equation*** is given by:

$$r(r-1) + \tfrac{1}{2}r = r\left(r - \tfrac{1}{2}\right) = 0,$$

so $r_1 = 0$ and $r_2 = \frac{1}{2}$.

Distinct Roots, $r_1 - r_2 \neq N$ 3

Example: We multiply our example by x and continue with

$$4x^2y'' + 2xy' + xy = 0,$$

trying a solution of the form: $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Differentiating y and entering into the equation gives:

$$4\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n} + 2\sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \sum_{n=1}^{\infty} a_{n-1}x^{r+n} = 0,$$

shifting the last index to match powers of x.

Note that when $n = 0$, we have

$$a_0\left[4r(r-1) + 2r\right] = 2a_0r(2r-1) = 0,$$

which is an alternate way to obtain the *indicial equation*.

Distinct Roots, $r_1 - r_2 \neq N$ 4

Example: For $n \geq 1$, we match powers of x, so

$$\left[4(r+n)(r+n-1) + 2(r+n)\right]a_n + a_{n-1} = 0.$$

From this we obtain the *recurrence relation*:

$$a_{n+1} = \frac{-a_n}{(2n+2r+2)(2n+2r+1)}, \qquad \text{for} \quad n = 0, 1, \dots$$

First Solution: Let $r = r_1 = 0$, then the *recurrence relation* becomes:

$$a_{n+1} = \frac{-a_n}{(2n+2)(2n+1)}, \qquad \text{for} \quad n = 0, 1, \dots,$$

so

$$a_1 = -\frac{a_0}{2\cdot 1}, \qquad a_2 = -\frac{a_1}{4\cdot 3} = \frac{a_0}{4!}, \qquad a_3 = -\frac{a_2}{6\cdot 5} = -\frac{a_0}{6!}.$$

Thus,

$$a_n = \frac{(-1)^n}{(2n)!}a_0.$$

Distinct Roots, $r_1 - r_2 \neq N$ 5

Example: Thus, the first solution is:

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^n.$$

Second Solution: Let $r = r_2 = \frac{1}{2}$, then the *recurrence relation* becomes:

$$b_{n+1} = \frac{-b_n}{(2n+3)(2n+2)}, \qquad \text{for} \quad n = 0, 1, \dots,$$

so

$$b_1 = -\frac{b_0}{3\cdot 2}, \qquad b_2 = -\frac{b_1}{5\cdot 4} = \frac{b_0}{5!}, \qquad b_3 = -\frac{b_2}{7\cdot 6} = -\frac{b_0}{7!}.$$

Thus,

$$b_n = \frac{(-1)^n}{(2n+1)!}b_0.$$

Thus, the second solution is:

$$y_2(x) = b_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{n+\frac{1}{2}}.$$

Distinct Roots, $r_1 - r_2 \neq N$ 6

Example: The series we see as solutions to this problem have similarities with series for **cosine** and **sine**.

Specifically, a change of variables gives:

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^n = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^{\frac{1}{2}})^{2n} = a_0 \cos\left(\sqrt{x}\right).$$

Similarly, the second linearly independent solution satisfies:

$$y_2(x) = b_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{n+\frac{1}{2}} = b_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^{\frac{1}{2}})^{2n+1} = b_0 \sin\left(\sqrt{x}\right).$$

Thus, we could write the general solution as

$$y(x) = a_0 \cos\left(\sqrt{x}\right) + b_0 \sin\left(\sqrt{x}\right),$$

which can readily be shown satisfies the ***ODE*** in this example:

$$4x^2y'' + 2xy' + xy = 0.$$

Repeated Roots, $r_1 = r_2 = r$ 1

Case 2. Repeated roots, $r_1 = r_2 = r$.

For this case, a basis for the solution of the *regular singular problem* satisfies:

$$y_1(x) = x^r(a_0 + a_1x + \dots + a_nx^n + \dots) = x^r \sum_{n=0}^{\infty} a_nx^n$$

and

$$y_2(x) = y_1(x)\ln(x) + x^r(b_1x + \dots + b_nx^n + \dots) = y_1(x)\ln(x) + x^r \sum_{n=1}^{\infty} b_nx^n$$

with these solutions converging for at least $|x| < \rho$, where ρ is the radius of convergence for $p(x)$ and $q(x)$.

Repeated Roots, $r_1 = r_2 = r$ 2

The form of the second solution is found in a manner similar to solving the *Cauchy-Euler equation*.

The first solution is found as before with:

$$y_1(x) = \phi(r,x) = x^r \sum_{n=0}^{\infty} a_n(r)x^n,$$

where the coefficients $a_n(r)$ are determined by a *recurrence relation* with the values of r found from the *indicial equation*

$$F(r) = r(r-1) + p_0r + q_0 = 0.$$

Our *regular singular problem* was $L[y] = x^2y'' + xp(x)y' + q(x)y = 0$, which with our first solution and the power series for $p(x)$ and $q(x)$ gives

$$L[\phi](r,x) = x^ra_0F(r) + \sum_{n=1}^{\infty}\left[a_nF(r+n) + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}]\right]x^{r+n} = 0,$$

where the second sum comes from multiplying the infinite series and collecting terms.

Repeated Roots, $r_1 = r_2 = r$ 3

Assuming $F(r+n) \neq 0$, the *recurrence relation* for the coefficients as a function of r satisfies:

$$a_n(r) = -\frac{\sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}]}{F(r+n)}, \qquad n \geq 1.$$

Selecting these coefficients reduces our power series solution to:

$$L[\phi](r,x) = x^ra_0F(r),$$

where $F(r) = (r-r_1)^2$ for our repeated root, so $L[\phi](r_1,x) = 0$, since our first solution is:

$$y_1(x) = \phi(r_1,x) = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n,$$

Significantly, we have

$$L\left[\frac{\partial\phi}{\partial r}\right](r_1,x) = a_0\,\frac{\partial}{\partial r}[x^r(r-r_1)^2]\Big|_{r=r_1} = a_0\,\left[(r-r_1)^2x^r\ln(x) + 2(r-r_1)x^r\right]\Big|_{r=r_1} = 0,$$

so $\frac{\partial\phi}{\partial r}(r_1,x)$ is also a solution to our problem.

Repeated Roots, $r_1 = r_2 = r$ 4

Since $\frac{\partial\phi}{\partial r}(r_1,x)$ is a solution and our first solution is:

$$y_1(x) = \phi(r_1,x) = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n,$$

we obtain the second solution:

$$\begin{aligned} y_2(x) &= \frac{\partial\phi(r,x)}{\partial r}\Big|_{r=r_1} = \frac{\partial}{\partial r}\left[x^r\sum_{n=0}^{\infty} a_n(r)x^n\right]\Bigg|_{r=r_1} \\ &= (x^{r_1}\ln(x))\sum_{n=0}^{\infty} a_n(r_1)x^n + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n \\ &= y_1(x)\ln(x) + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n, \qquad x > 0, \end{aligned}$$

where

$$a_n(r) = -\frac{\sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}]}{F(r+n)}, \qquad n \geq 1.$$

Bessel's Equation Order Zero 1

Bessel's Equation Order Zero satisfies:

$$x^2y'' + xy' + x^2y = 0,$$

where $x = 0$ is a ***regular singular point***.

From our definitions before we have $p(x) = 1$ and $q(x) = x^2$, which implies that $p_0 = 1$ and $q_0 = 0$.

Since p and q have convergent power series for all x, the solutions will converge for $|x| < \infty$.

The ***indicial equation*** is given by:

$$r(r-1) + r = r^2 = 0,$$

so $r_1 = r_2 = 0$, ***repeated root***.

Bessel's Equation Order Zero 2

With ***Bessel's equation order zero***,

$$x^2y'' + xy' + x^2y = 0,$$

we try a solution of the form: $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Differentiating y and entering into the equation gives:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0,$$

shifting the last index to match powers of x.

From the same powers of x, $(n+r)^2 a_n + a_{n-2} = 0$, which for $r = 0$ gives the ***recurrence relation***:

$$a_n = \frac{-a_{n-2}}{n^2}, \qquad \text{for} \quad n = 2, 3, 4, \dots,$$

with a_0 arbitrary and $a_1 = 0$.

Bessel's Equation Order Zero 3

The ***recurrence relation*** shows that the **odd powers** of x all vanish.

Letting $n = 2m$ in the ***recurrence relation*** gives:

$$a_{2m} = \frac{-a_{2m-2}}{(2m)^2}, \qquad \text{for} \quad n = 1, 2, 3, \dots,$$

so

$$a_2 = -\frac{a_0}{2^2}, \quad a_4 = -\frac{a_2}{(2\cdot 2)^2} = \frac{a_0}{2^4\cdot 2^2}, \quad a_6 = -\frac{a_4}{(2\cdot 3)^2} = -\frac{a_0}{2^6(3\cdot 2)^2}, \quad \dots$$

In general, we have

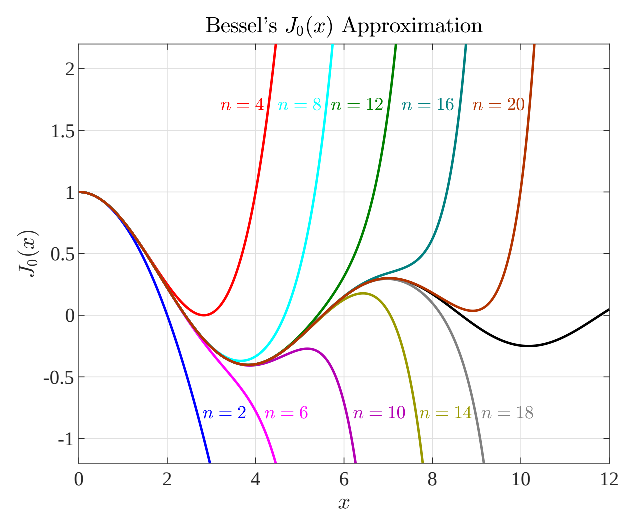
$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \dots$$

The ***first solution*** becomes:

$$y_1(x) = a_0 J_0(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}.$$

Bessel's Equation Order Zero 4

This ***first solution*** gives the ***Bessel function of the first kind of order zero***, $J_0(x)$. Below shows some polynomial approximations from the partial sums of the series solution.



Bessel's Equation Order Zero 5

Since the *Bessel's equation order zero* has only the **repeated root** $r = 0$ from the *indicial equation*, the second solution has the form:

$$y_2(x) = y_1(x)\ln(x) + x^r \sum_{n=1}^{\infty} b_n x^n = \ln(x)J_0(x) + \sum_{n=1}^{\infty} b_n x^n.$$

One technique to solve for this second solution is to substitute into *Bessel's equation* and solve for the coefficients, b_n.

Alternately, we use our results in deriving this form of the second solution, where we found that the coefficients satisfied:

$$b_n = a_n'(r), \qquad \text{where} \quad a_n(r) = -\frac{a_{n-2}(r)}{(n+r)^2},$$

evaluated at $r = 0$ based on the *recurrence relation* for coefficients of the first solution.

Bessel's Equation Order Zero 6

From the formula deriving the *recurrence relation*, we find $(r+1)^2 a_1(r) = 0$, so not only $a_1(0) = 0$, but $a_1'(0) = 0$.

It follows from the *recurrence relation* that

$$a_3'(0) = a_5'(0) = \cdots = a_{2k+1}'(0) = \cdots = 0.$$

The *recurrence relation* gives:

$$a_{2m}(r) = -\frac{a_{2m-2}(r)}{(2m+r)^2}, \qquad m = 1, 2, 3, \dots$$

Hence,

$$\begin{aligned} a_2(r) &= -\frac{a_0}{(2+r)^2}, \\ a_4(r) &= -\frac{a_2(r)}{(4+r)^2} = \frac{a_0}{(4+r)^2(2+r)^2}, \\ a_6(r) &= -\frac{a_4(r)}{(6+r)^2} = \frac{a_0}{(6+r)^2(4+r)^2(2+r)^2}, \end{aligned}$$

$$a_{2m}(r) = \frac{(-1)^m a_0}{(2m+r)^2(2m-2+r)^2 \cdots (4+r)^2(2+r)^2}, \qquad m = 1, 2, 3, \dots$$

Bessel's Equation Order Zero 7

Note that if

$$f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}\cdots(x-\alpha_n)^{\beta_n}.$$

then

$$\begin{aligned} f'(x) &= \beta_1(x-\alpha_1)^{\beta_1-1}\left[(x-\alpha_2)^{\beta_2}\cdots(x-\alpha_n)^{\beta_n}\right] \\ &\quad + \beta_2(x-\alpha_2)^{\beta_2-1}\left[(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_n)^{\beta_n}\right] + \cdots, \end{aligned}$$

Hence, for $x \neq \alpha_1, \alpha_2, \dots$

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2} + \cdots + \frac{\beta_n}{x-\alpha_n}.$$

Thus,

$$\frac{a_{2m}'(r)}{a_{2m}(r)} = -2\left(\frac{1}{2m+r} + \frac{1}{2m-2+r} + \cdots + \frac{1}{2+r}\right),$$

or with $r = 0$

$$a_{2m}'(0) = -2\left(\frac{1}{2m} + \frac{1}{2(m-1)} + \cdots + \frac{1}{2}\right) a_{2m}(0).$$

Bessel's Equation Order Zero 8

Define

$$H_m = \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{2} + 1,$$

then using this with the *recurrence relation*

$$a_{2m}'(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \dots$$

It follows that the second solution of *Bessel's equation order zero* (with $a_0 = 1$) satisfies:

$$y_2(x) = J_0(x)\ln(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \qquad x > 0.$$

Usually the second solution is taken to be the *Bessel function of the second kind of order zero*, which is defined as

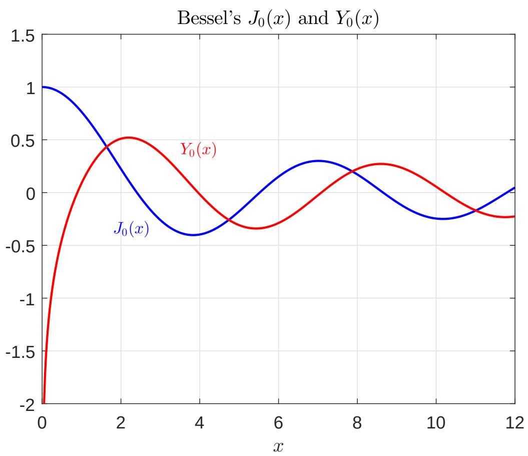
$$Y_0(x) = \tfrac{2}{\pi}\left[y_2(x) + (\gamma - \ln(2))J_0(x)\right],$$

where γ is the Euler-Máscheroni constant

$$\gamma = \lim_{n\to\infty}(H_n - \ln(n)) \approx 0.5772.$$

Bessel's Equation Order Zero

The standard solutions for ***Bessel's equation*** are the ***Bessel function of the first kind of order zero***, $J_0(x)$ and ***Bessel function of the second kind of order zero***, $Y_0(x)$.



Roots Differing by an Integer, $r_1 - r_2 = N$

Case 3. Roots Differing by an Integer, $r_1 - r_2 = N$, where N is a positive integer.

As before, one solution of the ***regular singular problem*** satisfies:

$$y_1(x) = |x|^{r_1} \sum_{n=0}^{\infty} a_n x^n.$$

The second linearly independent solution has the form:

$$y_2(x) = k\, y_1(x) \ln|x| + |x|^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

with these solutions converging for at least $|x| < \rho$, where ρ is the radius of convergence for $p(x)$ and $q(x)$.

This case divides into two subcases, depending on whether or not the logarithmic term appears, as k may be **zero**.

Roots Differing by an Integer, $r_1 - r_2 = N$

Case 3. Roots Differing by an Integer, $r_1 - r_2 = N$, where N is a positive integer.

This case is more complicated with the coefficients in the second solution satisfying:

$$b_n(r_2) = \frac{d}{dr}\left[(r - r_2)a_n(r)\right|_{r=r_2}, \qquad n = 0, 1, 2, \dots$$

with $a_0 = r - r_2$ and

$$k = \lim_{r \to r_2} (r - r_2)a_N(r).$$

In practice the best way to determine if $k = 0$ is to compute $a_n(r_2)$ and see if one finds $a_N(r_2)$.

If this is possible, then the second solution is readily found without the logarithmic term; otherwise, the logarithmic term must be included.

Example: Roots, $r_1 - r_2 = N$

Example: Consider the ***ODE***:

$$x^2y'' + 3xy' + 4x^4y = 0,$$

where $x = 0$ is a ***regular singular point***.

We try a solution of the form: $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Differentiating y and entering into the equation gives:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + 3\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + 4\sum_{n=4}^{\infty} a_{n-4}x^{n+r} = 0,$$

shifting the last index to match powers of x.

Examining this equation with $n = 0$ gives the ***indicial equation***:

$$F(r) = r(r-1) + 3r = r(r+2) = 0,$$

which has the roots $r_1 = 0$ and $r_2 = -2$ ($r_1 - r_2 = 2$).

Example: Roots, $r_1 - r_2 = N$ 2

Example: The series above could be rearranged in the following form:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r+2)x^{n+r} = -4\sum_{n=4}^{\infty} a_{n-4}x^{n+r}.$$

With $r_1 = 0$, we obtain the *recurrence relations*:

$$a_1(r_1+1)(r_1+3) = 3a_1 = 0, \quad a_2(r_1+2)(r_1+4) = 8a_2 = 0, \quad a_3(r_1+3)(r_1+5) = 15a_3 = 0,$$

$a_1 = a_2 = a_3 = 0$, and

$$a_n(r) = -\frac{4}{(n+r)(n+r+2)}a_{n-4}(r) \qquad \text{or} \qquad a_n = -\frac{4}{n(n+2)}a_{n-4}.$$

It follows that

$$a_4 = -\frac{4}{6\cdot 4}a_0 = -\frac{a_0}{3\cdot 2}, \quad a_8 = -\frac{4}{10\cdot 8}a_4 = \frac{a_0}{5!}, \quad a_{12} = -\frac{4}{14\cdot 12}a_8 = -\frac{a_0}{7!},$$

and $a_1 = a_5 = ... = a_{4n+1} = 0$, $a_2 = a_6 = ... = a_{4n+2} = 0$, and $a_3 = a_7 = ... = a_{4n+3} = 0$.

Example: Roots, $r_1 - r_2 = N$ 3

Example: The results above are combined to give the 1^{st} ***solution***:

$$y_1(x) = a_0\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{4m}.$$

To find the 2^{nd} ***solution*** we need to know if the ***logarithmic term*** needs to be included.

This term is unnecessary if

$$\lim_{r\to r_2} a_N(r) \quad \text{exists.}$$

For this example, $r_2 = -2$ and $N = 2$, so we examine

$$\lim_{r\to -2} a_2(r) = \frac{0}{(r+2)(r+4)} = 0,$$

which implies the second series may be computed directly with no ***logarithmic term***.

Thus, we try a solution of the form:

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2},$$

Example: Roots, $r_1 - r_2 = N$ 4

Example: Following the same process as finding $y_1(x)$, we have:

$$\sum_{n=0}^{\infty} b_n(n+r_2)(n+r_2+2)x^{n+r_2} = -4\sum_{n=4}^{\infty} b_{n-4}x^{n+r_2}.$$

With $r_2 = -2$, we obtain the *recurrence relations*:

$$b_1(r_2+1)(r_2+3) = -b_1 = 0, \quad b_2(r_2+2)(r_2+4) = 0b_2 = 0, \quad b_3(r_2+3)(r_2+5) = 3b_3 = 0,$$

$b_1 = b_3 = 0$ (b_2 is arbitrary and generates y_1, so take $b_2 = 0$), and

$$b_n(r) = -\frac{4}{(n+r)(n+r+2)}b_{n-4}(r) \qquad \text{or} \qquad b_n = -\frac{4}{n(n-2)}b_{n-4}.$$

It follows that

$$b_4 = -\frac{4}{4\cdot 2}b_0 = -\frac{b_0}{2\cdot 1}, \quad b_8 = -\frac{4}{8\cdot 6}b_4 = \frac{b_0}{4!}, \quad b_{12} = -\frac{4}{12\cdot 10}b_8 = -\frac{b_0}{6!},$$

and $b_1 = b_5 = ... = b_{4n+1} = 0$, $b_2 = b_6 = ... = b_{4n+2} = 0$, and $b_3 = b_7 = ... = b_{4n+3} = 0$.

Example: Roots, $r_1 - r_2 = N$ 5

Example: These results are combined to give the 2^{nd} ***solution***:

$$y_2(x) = b_0x^{-2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{4m}.$$

It follows that our general solution for this example is:

$$y(x) = a_0\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{4m} + b_0x^{-2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{4m}.$$

This could be rewritten:

$$\begin{aligned} y(x) &= a_0x^{-2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}(x^2)^{2m+1} + b_0x^{-2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}(x^2)^{2m}, \\ &= x^{-2}\big(a_0\sin(x^2) + b_0\cos(x^2)\big). \end{aligned}$$

Example 2: Roots, $r_1 - r_2 = N$ 1

Example 2: Consider the ***ODE***:

$$x^2 y'' - xy = 0,$$

where $x = 0$ is a ***regular singular point***.

Try a solution of the form: $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

Differentiating y and entering into the equation gives:

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0,$$

shifting the last index to match powers of x.

Examining this equation with $n = 0$ gives the ***indicial equation***:

$$F(r) = r(r-1) = 0,$$

which has the roots $r_1 = 1$ and $r_2 = 0$ $(r_1 - r_2 = 1)$.

Example 2: Roots, $r_1 - r_2 = N$ 2

Example 2: The series above is rearranged in the following form:

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}.$$

With $r_1 = 1$, we obtain the ***recurrence relation***:

$$a_n(r) = \frac{1}{(n+r)(n+r-1)} a_{n-1}(r) \quad \text{or} \quad a_n = \frac{1}{n(n+1)} a_{n-1}, \quad n = 1, 2, 3, \dots$$

It follows that

$$a_1 = \frac{1}{1 \cdot 2} a_0, \quad a_2 = \frac{1}{2 \cdot 3} a_1 = \frac{a_0}{2!3!}, \quad a_3 = \frac{1}{3 \cdot 4} a_2 = \frac{a_0}{3!4!},$$

so

$$a_n = \frac{a_0}{n!(n+1)!}, \qquad n = 1, 2, 3, \dots$$

Example 2: Roots, $r_1 - r_2 = N$ 3

Example 2: The results above are combined to give the 1^{st} ***solution***:

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n+1}.$$

To find the 2^{nd} ***solution*** we need to know if the ***logarithmic term*** needs to be included.

This term is necessary if

$$\lim_{r \to r_2} a_N(r) \quad \text{fails to exist.}$$

For this example, $r_2 = 0$ and $N = 1$, so we examine

$$\lim_{r \to 0} a_1(r) = \frac{a_0(r)}{(r+1)r}.$$

Since a_0 is arbitrary (non-zero), this limit is undefined, so a second series solution requires the ***logarithmic term***.

For $r_2 = 0$, we try a solution of the form:

$$y_2(x) = k\, y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2},$$

Example 2: Roots, $r_1 - r_2 = N$ 4

Example 2: Insert $y_2(x) = k\, y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^n$ into the ***ODE***, so

$$\begin{aligned} x^2 \left[k y_1'' \ln(x) + 2k y_1' \tfrac{1}{x} - k y_1 \tfrac{1}{x^2} + \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2} \right] & \\ - kxy_1 \ln(x) - \sum_{n=0}^{\infty} b_n x^{n+1} &= 0. \end{aligned}$$

Because $y_1(x)$ is a solution of the ***ODE***, $k \ln(x)[x^2 y_1'' - xy_1] = 0$, which reduces this expression to

$$2kxy_1' - ky_1 + \sum_{n=2}^{\infty} n(n-1) b_n x^n - \sum_{n=0}^{\infty} b_n x^{n+1} = 0.$$

Using the series solution for $y_1(x)$ with $a_0 = 1$ and shifting indices, we obtain

$$\sum_{n=0}^{\infty} \left(\tfrac{2k}{(n!)^2} - \tfrac{k}{n!(n+1)!} \right) x^{n+1} + \sum_{n=1}^{\infty} b_{n+1}(n+1)n x^{n+1} - \sum_{n=0}^{\infty} b_n x^{n+1} = 0.$$

Example 2: Roots, $r_1 - r_2 = N$ 5

Example 2: From the series above our *recurrence relation* gives:

$$k - b_0 = 0, \qquad \text{or} \qquad k = b_0$$

and

$$\frac{2k}{(n!)^2} - \frac{k}{n!(n+1)!} + n(n+1)b_{n+1} - b_n = 0, \qquad n = 1, 2, 3, \dots$$

Equivalently,

$$b_{n+1} = \frac{1}{n(n+1)} \left[b_n - \frac{(2n+1)k}{n!(n+1)!} \right], \qquad n = 1, 2, 3, \dots$$

For convenience we take $a_0 = 1$ and $b_0 = k = 1$.

The constant b_1 is still arbitrary (as it would generate $y_1(x)$ again), so we select $b_1 = 0$ and find a particular 2^{nd} *solution* using the *recurrence relation*:

$$y_2(x) = y_1(x)\ln(x) + 1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 - \frac{35}{1728}x^4 - \dots$$