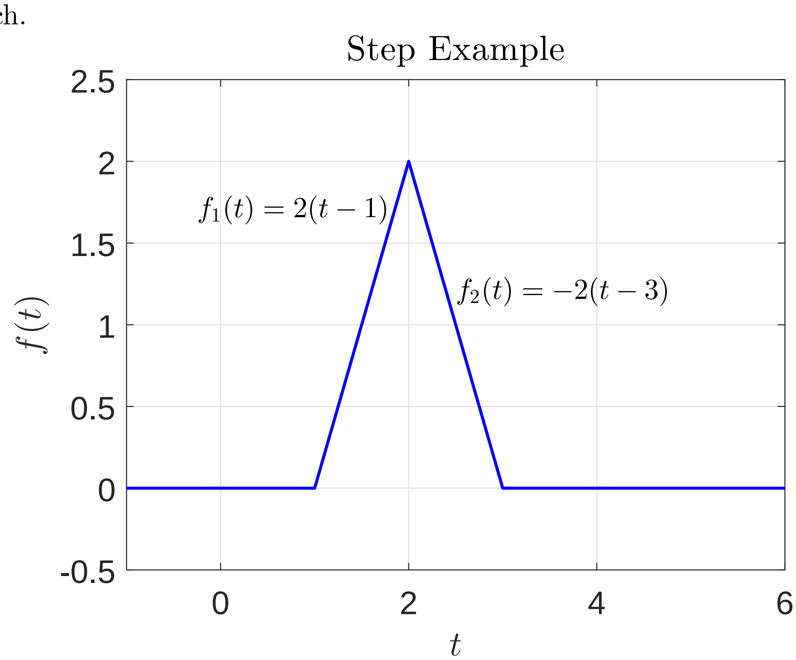
Additional Notes for Step Function

A more detailed study is provided for the **step function** and its **Laplace transform**. We consider the function in the figure below and want to express this function in terms of step functions. It is important to remember that the **step function** is an approximation of an **"on/off"** switch.



The **unit step function** is defined:

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

Thus, this function is **"off"** until $t = c$, then it turns **"on"**. In the figure above we see that $f(t)$ is **"off"** until $t = 1$, then $f_1(t)$ is turned **"on"**. At $t = 2$, $f_1(t)$ is turned **"off"**, and $f_2(t)$ is turned **"on"**. At $t = 3$, all components are turned **"off"**.

This can be readily expressed in terms of the unit step function (thinking as $+u_c(t)$ as turning something **"on"** at $t = c$ and $-u_c(t)$ as turning something **"off"** at $t = c$). From the description above, we have

$$\begin{aligned} f(t) &= f_1(t)u_1(t) - f_1(t)u_2(t) + f_2(t)u_2(t) - f_2(t)u_3(t), \\ &= f_1(t)\big(u_1(t) - u_2(t)\big) + f_2(t)\big(u_2(t) - u_3(t)\big), \\ &= 2(t-1)\big(u_1(t) - u_2(t)\big) - 2(t-3)\big(u_2(t) - u_3(t)\big). \end{aligned}$$

Laplace Transform for $f(t)$

We have the following important entry in the Laplace transform table from our theorem in the lecture notes.

$$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)] = e^{-cs}F(s), \qquad s > a. \tag{1}$$

It is important to note that the **phase shift** in f, $(f(t-c))$, matches the value c where the step function turns **"on"**.

To take the Laplace transform for our example

$$f(t) = 2(t-1)\big(u_1(t) - u_2(t)\big) - 2(t-3)\big(u_2(t) - u_3(t)\big),$$

we rearrange terms to have the same phase shifts, *i.e.*, modify $f_1(t)$ and $f_2(t)$ to align with the step functions in $f(t)$. We can rewrite

$$\begin{aligned} f(t) &= 2(t-1)u_1(t) - \big(2(t-2)+2\big)u_2(t) - \big(2(t-2)-2\big)u_2(t) + 2(t-3)u_3(t), \\ &= 2(t-1)u_1(t) - 4(t-2)u_2(t) + 2(t-3)u_3(t). \end{aligned}$$

Now we use the Laplace transform entry that

$$\mathcal{L}[t] = \frac{1}{s^2}, \qquad s > 0. \tag{2}$$

It easily follows from Eqns. (1) and (2) and the linearity of the Laplace tranform that

$$\mathcal{L}[f(t)] = 2\frac{e^{-s}}{s^2} - 4\frac{e^{-2s}}{s^2} + 2\frac{e^{-3s}}{s^2}, \qquad s > 0.$$