Additional Example for Step Function

Consider the IVP given by:

$$y'' + 2y' + y = \begin{cases} t, & 0 \le t < 2, \\ 0, & t \ge 2, \end{cases} = t - (t-2)u_2(t) - 2u_2(t), \qquad y(0) = 0, \quad y'(0) = 2.$$

Let $Y(s) = \mathcal{L}[y(t)]$, then

$$s^2 Y(s) - 2 + 2sY(s) + Y(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s},$$

or

$$Y(s) = \frac{2}{(s+1)^2} + \frac{1}{s^2(s+1)^2} - \frac{e^{-2s}}{s^2(s+1)^2} - \frac{2e^{-2s}}{s(s+1)^2}.$$

We need the PFD of

$$\frac{1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} \quad \text{and} \quad \frac{2}{s(s+1)^2} = \frac{E}{s} + \frac{F}{s+1} + \frac{G}{(s+1)^2}.$$

The first PFD examines

$$1 = As(s+1)^2 + B(s+1)^2 + Cs^2(s+1) + Ds^2.$$

With $s = 0$, we have $B = 1$, and with $s = -1$, we have $D = 1$. The cubic terms give $A + C = 0$, and the quadratic terms are $2A + B + C + D = 0$. Combining these gives $A = -2$ and $C = 2$. Similar computations give $E = 2$, $F = -2$, and $G = -2$. These results combine to give:

$$Y(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2} + e^{-2s}\left(-\frac{1}{s^2} + \frac{1}{(s+1)^2}\right).$$

It follows that

$$y(t) = -2 + t + 2e^{-t} + 3e^{-t} + u_2(t)\Big(- (t-2) + (t-2)e^{-(t-2)}\Big).$$