Math 337 - Elementary Differential Equations

Lecture Notes – Laplace Transforms: Part B

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Outline

Inverse Laplace Transforms

Theorem (Inverse Laplace Transform)

If $f(t)$ and $g(t)$ are piecewise continuous and have exponential order with exponent a on $[0,\infty)$ and $F = G$, where $F = \mathcal{L}[f]$ and $G = \mathcal{L}[g]$, then $f(t) = g(t)$ at all points where both f and g are continuous. In particular, f and g are continuous on $[0,\infty)$, then $f(t) = g(t)$ for all $t \in [0,\infty)$.

The functions may disagree at points of **discontinuity**

Definition (Inverse Laplace Transform)

If $f(t)$ is piecewise continuous and has exponential order with exponent a on $[0,\infty)$ and $\mathcal{L}[f(t)] = F(s)$, then we call f the **inverse Laplace transform** of F, and denote it by

$$f(t) = \mathcal{L}^{-1}[F(s)].$$

Linearity of Inverse Laplace Transforms

Theorem (Linearity of Inverse Laplace Transform)

Assume that $f_1 = \mathcal{L}^{-1}[F_1]$ and $f_2 = \mathcal{L}^{-1}[F_2]$ are piecewise continuous and has exponential of order with exponent a on $[0, \infty)$. Then for any constants c_1 and c_2,

$$\mathcal{L}^{-1}[c_1F_1 + c_2F_2] = c_1\mathcal{L}^{-1}[F_1] + c_2\mathcal{L}^{-1}[F_2] = c_1f_1 + c_2f_2.$$

Example: Find $\mathcal{L}^{-1}\left[\frac{2}{(s-3)^4} + \frac{12}{s^2+16} + \frac{5(s+2)}{s^2+4s+5}\right]$.

Rewrite as

$$\frac{1}{3}\mathcal{L}^{-1}\left[\frac{3!}{(s-3)^4}\right] + 3\mathcal{L}^{-1}\left[\frac{4}{s^2+16}\right] + 5\mathcal{L}^{-1}\left[\frac{(s+2)}{(s+2)^2+1}\right]$$

Laplace Table

With **Exponential Shift Theorem**

$$\frac{1}{3}e^{3t}t^3 + 3\sin(4t) + 5e^{-2t}\cos(t)$$

Linearity of Inverse Laplace Transforms

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$$\mathcal{L}^{-1}[c_1 F_1 + c_2 F_2] = c_1 \mathcal{L}^{-1}[F_1] + c_2 \mathcal{L}^{-1}[F_2] = c_1 f_1 + c_2 f_2.$$

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Laplace Table

With **Exponential Shift Theorem**

$$\frac{1}{3}e^{3t}t^3 + 3\sin(4t) + 5e^{-2t}\cos(t)$$

Example: DE with Laplace Transform 1

Example: Consider the **initial value problem**:

$$y'' + y = 10e^{-t}\cos(2t) \qquad \text{with} \quad y(0) = 2, \quad y'(0) = 1$$

Let $Y(s) = \mathcal{L}[y(t)]$, then taking **Laplace transforms** gives

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{10(s+1)}{(s+1)^2+4}$$

or

$$(s^2+1)Y(s) = 2s + 1 + \frac{10(s+1)}{(s+1)^2+4}$$

Equivalently,

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{10(s+1)}{(s^2+1)((s+1)^2+4)}$$

Example: DE with Laplace Transform 2

Example: Since

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{10(s+1)}{(s^2+1)((s+1)^2+4)},$$

and the first term is already in simplest form, **partial fractions decomposition** gives

$$\frac{10(s+1)}{(s^2+1)((s+1)^2+4)} = \frac{As+B}{s^2+1} + \frac{C(s+1)+D\cdot 2}{(s+1)^2+4}.$$

It follows that

$$10(s+1) = (As+B)(s^2+2s+5) + (C(s+1)+2D)(s^2+1)$$

Let $s = i$, then

$$10 + 10i = (B + Ai)(4 + 2i) = (4B - 2A) + i(4A + 2B)$$

Example: DE with Laplace Transform

Example: Equating the real and imaginary parts of the previous equation give:

$$-2A + 4B = 10 \qquad \text{and} \qquad 4A + 2B = 10$$

Solving the **linear equations** gives $A = 1$ and $B = 3$

Since

$$10(s+1) = (As+B)(s^2+2s+5) + (C(s+1)+2D)(s^2+1),$$

the cubic (s^3) terms give $0 = A + C$ or $C = -1$.

The constant terms give

$$10 = 5B + C + 2D \qquad \text{or} \qquad D = -2$$

Example: DE with Laplace Transform 4

Example: Thus,

$$\begin{aligned} Y(s) &= \frac{2s+1}{s^2+1} + \frac{10(s+1)}{(s^2+1)((s+1)^2+4)}, \\ Y(s) &= \frac{2s+1}{s^2+1} + \frac{s}{s^2+1} + \frac{3}{s^2+1} - \frac{(s+1)}{(s+1)^2+4} - \frac{2\cdot 2}{(s+1)^2+4} \\ Y(s) &= \frac{3s}{s^2+1} + \frac{4}{s^2+1} - \frac{s+1}{(s+1)^2+4} - \frac{2\cdot 2}{(s+1)^2+4} \end{aligned}$$

Laplace Table

This last line allows easy application of the **Inverse Laplace Transform** to obtain the solution

$$y(t) = 3\cos(t) + 4\sin(t) - e^{-t}\cos(2t) - 2e^{-t}\sin(2t).$$

Example: DE with Laplace Transform

Example: Thus,

$$\begin{aligned}
Y(s) &= \frac{2s+1}{s^2+1} + \frac{10(s+1)}{(s^2+1)((s+1)^2+4)}, \\
Y(s) &= \frac{2s+1}{s^2+1} + \frac{s}{s^2+1} + \frac{3}{s^2+1} - \frac{(s+1)}{(s+1)^2+4} - \frac{2\cdot 2}{(s+1)^2+4} \\
Y(s) &= \frac{3s}{s^2+1} + \frac{4}{s^2+1} - \frac{s+1}{(s+1)^2+4} - \frac{2\cdot 2}{(s+1)^2+4}
\end{aligned}$$

Laplace Table

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$$y(t) = 3\cos(t) + 4\sin(t) - e^{-t}\cos(2t) - 2e^{-t}\sin(2t).$$

Example: Laplace Transform with Maple 1

Example: Consider the IVP:

$$y'' + 4y' + 13y = 36te^{-2t}\sin(3t), \qquad y(0) = -3 \quad \text{and} \quad y'(0) = 6.$$

It is easy to see that this would be a **messy** problem to work with techniques such as the ***Method of Undetermined Coefficients*** though easily solved with **Maple's** *dsolve*.

We use **Maple** to demonstrate how to obtain:

- More ***Laplace transform*** elements.
- Perform ***Partial Fractions Decompositions***.
- Take ***inverse Laplace transforms***, giving a solution.

A **Special Maple Sheet** is provided with details for the steps to solve this problem and some discussion of the special commands, many listed on the next slide.

Example: Laplace Transform with Maple 2

Maple commands for solving an ***IVP*** using ***Laplace transforms***:

- Enter special package for integral transforms:
  $with(inttrans)$:
- Enter differential equation and initial conditions:
  $de := diff(y(t), t\$2)+4 \cdot diff(y(t), t) + 13 \cdot y(t) = 36 \cdot t \cdot \exp(-2 \cdot t) \cdot \sin(3 \cdot t);$
  $y(0) := -3;\ \mathrm{D}(y)(0) := 6;$
- Find the ***Laplace transform*** of the ***IVP***:
  $soln := laplace(de, t, s);$
- Use **Maple's** algebra to find $Y(s)$:
  $soln1 := solve(soln, laplace(y(t), t, s));$
- Perform a ***Partial Fractions Decomposition***:
  $soln2 := convert(soln1, parfrac, s);$
- Find the ***inverse Laplace transform*** to obtain the solution:
  $invlaplace(soln2, t, s);$

As noted earlier, **Maple's** *dsolve* command solves this problem most quickly.

However, the commands above can help with individual steps for solving ***ODEs*** with ***Laplace transforms***.

Discontinuous Functions

Unit Step function or **Heaviside function** satisfies

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

The translated version of the **Unit Step function** by c units is

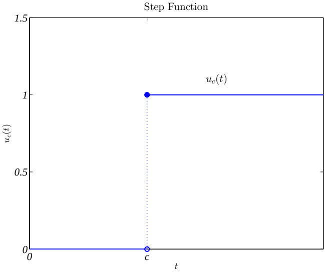
$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases}$$

which represents a switch **turning on** at $t = c$

An **indicator function**, which is **on** for $c \leq t < d$, satisfies

$$u_{cd}(t) = u_c(t) - u_d(t) = \begin{cases} 0, & t < c \quad \text{or} \quad t \geq d, \\ 1, & c \leq t < d, \end{cases}$$

Laplace Transform of Step Function



Laplace Transform of Step Function;

$$\mathcal{L}[u_c(t)] = \int_0^\infty e^{-st}u_c(t)dt = \int_c^\infty e^{-st}dt = \lim_{A\to\infty}\int_c^A e^{-st}dt$$
$$= \lim_{A\to\infty}\left(\frac{e^{-cs}}{s} - \frac{e^{-sA}}{s}\right) = \frac{e^{-cs}}{s}, \qquad s > 0$$

Laplace Transform of Step Function

Theorem

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geq 0$*, and if* c *is a nonnegative constant, then*

$$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)] = e^{-cs}F(s), \qquad s > a.$$

Conversely, if $f(t) = \mathcal{L}^{-1}[F(s)]$*, then*

$$u_c(t)f(t-c) = \mathcal{L}^{-1}[e^{-cs}F(s)].$$

This theorem states that the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}

Example with Step Function 1

Example with Step Function: Consider the following **initial value problem**:

$$y'' + 2y' + 5y = u_2(t) - u_5(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Take **Laplace transforms** and obtain

$$(s^2 + 2s + 5)Y(s) = \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}$$

This rearranges to

$$Y(s) = \frac{e^{-2s} - e^{-5s}}{s(s^2 + 2s + 5)}$$

Partial fraction decomposition gives

$$\frac{1}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{B(s + 1) + 2C}{(s + 1)^2 + 4}$$

Example with Step Function 2

Example with Step Function: Partial fraction decomposition gives

$$1 = A\left((s+1)^2 + 4\right) + \left(B(s+1) + 2C\right)s$$

With $s = 0$, $1 = 5A$ or $A = \frac{1}{5}$

The s^2 coefficient gives $0 = A + B$, so $B = -\frac{1}{5}$

The s^1 coefficient gives $0 = 2A + B + 2C$, so $C = -\frac{1}{10}$

Thus,

$$Y(s) = \left(\frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}(s+1) + \frac{2}{10}}{(s+1)^2 + 4}\right)\left(e^{-2s} - e^{-5s}\right)$$

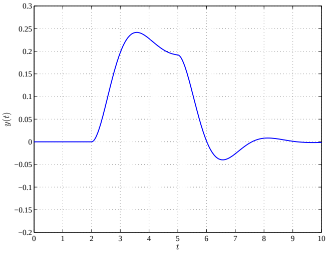
Example with Step Function 3

Example with Step Function: With the **Laplace transform**

$$Y(s) = \left(\frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}(s+1) + \frac{2}{10}}{(s+1)^2 + 4} \right) \left(e^{-2s} - e^{-5s}\right)$$

The theorem for step functions allows the **inverse Laplace transform** yielding

$$\begin{aligned} y(t) \quad &= \quad \frac{u_2(t)}{10}\left(2 - 2e^{-(t-2)}\cos(2(t-2)) - e^{-(t-2)}\sin(2(t-2))\right) \\ &\quad - \frac{u_5(t)}{10}\left(2 - 2e^{-(t-5)}\cos(2(t-5)) - e^{-(t-5)}\sin(2(t-5))\right) \end{aligned}$$



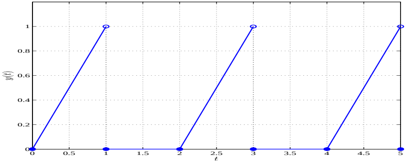
Periodic Functions

Definition

A function f is said to be **periodic with period $T > 0$** if

$$f(t+T) = f(t)$$

for all t in the domain of f.



A **sawtooth** waveform

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < 2. \end{cases}$$

and $f(t)$ has period 2

Periodic and Window functions

Consider a **periodic function** $f(t)$. Define the **window function**, $f_T(t)$, as follows:

$$f_T(t) = f(t)\left[1 - u_T(t)\right] = \begin{cases} f(t), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The **Laplace transform** $F_T(s)$ satisfies:

$$F_T(s) = \int_0^\infty e^{-st} f_T(t)dt = \int_0^T e^{-st} f_T(t)dt.$$

The **window function** specifies values of $f(t)$ over a single period

This can be replicated k periods to the right as

$$f_T(t - kT)u_{kT}(t) = \begin{cases} f(t - kT), & kT \le t \le (k+1)T \\ 0, & \text{otherwise} \end{cases}$$

Laplace for Periodic Functions

By summing n time shifted replications of the **window function**, $f_T(t-kT)u_{kT}(t)$, $k=0,...,n-1$, gives $f_{nT}(t)$, the periodic extension of $f_T(t)$ to the interval $[0,nT]$,

$$f_{nT}(t)=\sum_{k=0}^{n-1}f_T(t-kT)u_{kT}(t)$$

Theorem

If f is periodic with period T and is piecewise continuous on $[0,T]$, then

$$\mathcal{L}[f(t)]=\frac{F_T(s)}{1-e^{-sT}}=\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}.$$

Laplace for Periodic Functions

Proof: From our earlier theorem, we have for each $k \geq 0$,

$$\mathcal{L}[f_T(t-kT)u_{kT}(t)] = e^{-kTs}\mathcal{L}[f_T(t)] = e^{-kTs}F_T(s).$$

By linearity of $\mathcal{L}$, the **Laplace transform** of f_{nT} is

$$\begin{aligned} F_{nT}(s) &= \int_0^{nT} e^{-st}f(t)dt = \sum_{k=0}^{n-1}\mathcal{L}[f_T(t-kT)u_{kT}(t)] \\ &= \sum_{k=0}^{n-1} e^{-kTs}F_T(s) = F_T(s)\sum_{k=0}^{n-1}\left(e^{-Ts}\right)^k = F_T(s)\frac{1-\left(e^{-Ts}\right)^n}{1-e^{-sT}}. \end{aligned}$$

The last term comes from summing a geometric series. With $e^{-sT} < 1$,

$$F(s) = \lim_{n->\infty}\int_0^{nT} e^{-st}dt = \lim_{n->\infty} F_T(s)\frac{1-\left(e^{-Ts}\right)^n}{1-e^{-sT}} = \frac{F_T(s)}{1-e^{-sT}}$$

Sawtooth Function

Return to **sawtooth** waveform

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < 2. \end{cases} \qquad \text{and } f(t) \text{ has period 2}$$

The **theorem** for the **Laplace transform** of **periodic function** gives

$$\mathcal{L}[f(t)] = \frac{\int_0^2 e^{-st} f(t)dt}{1 - e^{-2s}}$$

But

$$\int_0^2 e^{-st} f(t)dt = \int_0^1 te^{-st}dt = \frac{1 - se^{-s} - e^{-s}}{s^2},$$

so

$$\mathcal{L}[f(t)] = \frac{1 - se^{-s} - e^{-s}}{s^2\left(1 - e^{-2s}\right)}$$

IVP with Periodic Forcing Function 1

Example: Consider the following initial value problem:

$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

with the **square** waveform as the **periodic forcing function**:

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2. \end{cases} \qquad \text{and } f(t) \text{ has period } 2$$

The **theorem** for the **Laplace transform** of **square** waveform gives

$$\mathcal{L}[f(t)] = \frac{\int_0^2 e^{-st} f(t)dt}{1 - e^{-2s}}$$

But

$$\int_0^2 e^{-st} f(t)dt = \int_0^1 e^{-st} dt = \frac{1 - e^{-s}}{s},$$

so

$$\mathcal{L}[f(t)] = \frac{1 - e^{-s}}{s\left(1 - e^{-2s}\right)} = \frac{1}{s\left(1 + e^{-s}\right)}$$

IVP with Periodic Forcing Function 2

Example: Taking the **Laplace transform** of the **IVP** with $\mathcal{L}[y(t)] = Y(s)$, we have:

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s(1+e^{-s})}$$

Thus,

$$Y(s) = \frac{1}{s(s^2+4)(1+e^{-s})}$$

Partial fractions decomposition gives

$$\frac{1}{s(s^2+4)} = \frac{1/4}{s} - \frac{s/4}{s^2+4},$$

while

$$\frac{1}{1+e^{-s}} = \frac{1}{1-(-e^{-s})} = 1 - e^{-s} + e^{-2s} - ... + (-1)^n e^{-ns} +$$

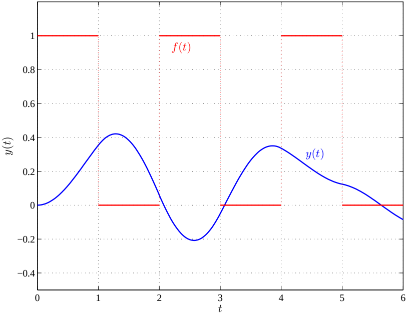
IVP with Periodic Forcing Function 3

Example: So

$$Y(s) = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\sum_{k=0}^{\infty}(-1)^k e^{-ks}$$

Taking the **inverse Laplace transform** gives:

$$y(t) = \frac{1}{4}\left(1-\cos(2t)\right) + \frac{1}{4}\sum_{k=1}^{\infty}(-1)^k u_k(t)\left(1-\cos(2(t-k))\right)$$



Impulse Function

Impulse Function: Some applications have phenomena of an **impulsive nature**, *e.g.*, a large magnitude force over a very short time

$$ay'' + by' + cy = g(t),$$

where $g(t)$ is very large for $t \in [t_0, t_0 + \varepsilon)$ and is otherwise zero

Example: Let $t_0 = 0$ be a real number and ε be a small positive constant

Suppose $t_0 = 0$ and $g(t) = I_0 \delta_\varepsilon(t)$, where

$$\delta_\varepsilon(t) = \frac{u_0(t) - u_\varepsilon(t)}{\varepsilon} = \begin{cases} \frac{1}{\varepsilon}, & 0 \le t < \varepsilon, \\ 0, & t < 0 \quad \text{or} \quad t \ge \varepsilon. \end{cases}$$

Consider the **mass-spring** system $m = 1$, $\gamma = 0$, and $k = 1$

$$y'' + y = I_0 \delta_\varepsilon(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Impulse Function - Example

Example: With $\delta_\varepsilon(t) = \frac{u_0(t)-u_\varepsilon}{\varepsilon}$, the **Laplace transform** is easy for

$$y'' + y = I_0\delta_\varepsilon(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

It satisfies

$$(s^2+1)Y(s) = \frac{I_0}{\varepsilon}\left(\frac{1-e^{-\varepsilon s}}{s}\right),$$

so

$$Y(s) = \frac{I_0}{\varepsilon}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\left(1-e^{-\varepsilon s}\right)$$

The **inverse Laplace transform** gives

$$y_\varepsilon(t) = \frac{I_0}{\varepsilon}\left(u_0(t)(1-\cos(t)) - u_\varepsilon(t)(1-\cos(t-\varepsilon))\right)$$

Impulse Function - Example

Example: Since

$$y_\varepsilon(t) = \frac{I_0}{\varepsilon}\left(u_0(t)(1-\cos(t)) - u_\varepsilon(t)(1-\cos(t-\varepsilon))\right),$$

equivalently:

$$y_\varepsilon(t) = \begin{cases} 0, & t<0, \\ \frac{I_0}{\varepsilon}(1-\cos(t)) & 0 \le t < \varepsilon, \\ \frac{I_0}{\varepsilon}(\cos(t-\varepsilon)-\cos(t)) & t \ge \varepsilon. \end{cases}$$

The limiting case is

$$y_0(t) = \lim_{\varepsilon\to 0} y_\varepsilon(t) = u_0(t) I_0 \sin(t) = \begin{cases} 0, & t<0, \\ I_0\sin(t), & t \ge 0. \end{cases}$$

Unit Impulse Function

Unit Impulse Function: Rather than using the definition of $\delta_\varepsilon(t-t_0)$ to model an impulse, then take the limit as $\varepsilon \to 0$, we define an idealized **unit Impulse Function,** δ

- The "function" δ imparts an impulse of magnitude **1** at $t = t_0$, but is **zero** for all other values of t
- **Properties of** $\delta(t-t_0)$
  1. Limiting behavior:
  $$\delta(t-t_0) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t-t_0) = 0$$
  2. If f is continuous for $t \in [a,b)$ and $t_0 \in [a,b)$, then
  $$\int_a^b f(t)\delta(t-t_0)dt = \lim_{\varepsilon\to 0}\int_a^b f(t)\delta_\varepsilon(t-t_0)dt = f(t_0).$$

$\delta(t-t_0)$

Dirac delta function, $\delta(t-t_0)$: This is not an ordinary function in elementary calculus, and it satisfies:

$$\int_a^b \delta(t-t_0)dt = \begin{cases} 1, & \text{if} \quad t_0 \in [a,b), \\ 0, & \text{if} \quad t_0 \notin [a,b). \end{cases}$$

The **Laplace transform of** $\delta(t-t_0)$ follows easily:

$$\mathcal{L}[\delta(t-t_0)] = \int_0^\infty e^{-st}\delta(t-t_0)dt = e^{-st_0}$$

Note: $\mathcal{L}[\delta(t)] = 1$.

The **delta function** is the **symbolic derivative** of the **Heaviside function**, so

$$\delta(t-t_0) = u'(t-t_0)$$

This is rigorously true in the theory of **generalized functions** or **distributions**

Example for $\delta(t - t_0)$ 1

Example: Consider the initial value problem:

$$y'' + 2y' + 2y = \frac{t}{\pi}\delta(t - \pi), \qquad y(0) = 0, \quad y'(0) = 1$$

The **Laplace transform** of the forcing function is

$$F(s) = \int_0^\infty e^{-st}\left(\frac{t}{\pi}\delta(t - \pi)\right) dt = e^{-\pi s}$$

It follows that the **Laplace transform** of the IVP is

$$s^2 Y(s) - 1 + 2sY(s) + 2Y(s) = e^{-\pi s},$$

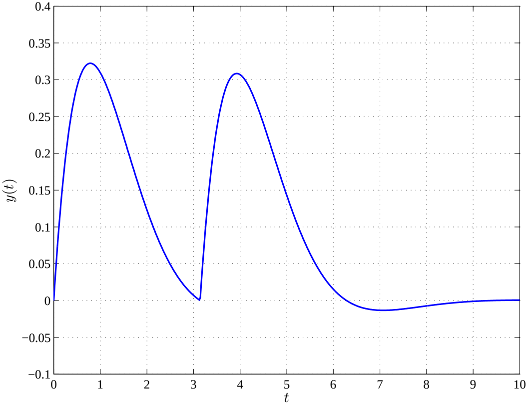
so

$$Y(s) = \frac{1 + e^{-\pi s}}{(s+1)^2 + 1}$$

Example for $\delta(t - t_0)$ 2

Example: Since $Y(s) = \frac{1+e^{-\pi s}}{(s+1)^2+1}$, the **inverse Laplace transform** satisfies:

$$y(t) = e^{-t}\sin(t) + u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$$



Laplace Table

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	t^n, $n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
6.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
7.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
8.	$t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
9.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
10.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$,
11.	$e^{ct}f(t)$	$F(s-c)$,
12.	$\delta(t-c)$,	e^{-cs},
13.	$t^n f(t)$,	$(-1)^n F^{(n)}(s)$,