Calculus for the Life Sciences

Lecture Notes – Separable Differential Equations

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Outline

Introduction

Introduction

- We have learned several methods of solving differential equations
  - Malthusian Growth and Radioactive decay - Solution recognition
  - Newton's Law of Cooling and Mixing Problems - Substitution to create above form
  - Time varying - Solved by integration
  - Numerical methods - Handles ones not solvable by other means
- This section examines a class of differential equations that separate into two integration problems for their solution

Malthusian Growth Model

Malthusian Growth Model has a simple exponential form for a solution

- A simple Malthusian growth model with a single growth rate is very limited in applications
- For discrete Malthusian growth models, a time varying component added to the model predicts the population much more accurately
- Time varying growth rates are very appropriate for human populations accounting for changes in growth rates due to changing societal conditions

Malthusian Growth Model for U. S. 1

Malthusian Growth Model for U. S. Consider constant growth model

$$\frac{dP(t)}{dt} = r\,P(t), \quad \text{with} \quad P(t_0) = P_0$$

- The U. S. census data gives the population as 3.93 million in 1790 and 248.7 million in 1990
- Let t be the time in years after 1790
- The solution to the Malthusian growth model is

$$P(t) = 3.93\,e^{rt}$$

- Since $P(200) = 248.7$, then $r = \left(\frac{1}{200}\right)\ln\left(\frac{248.7}{3.93}\right) = 0.02074$, so

$$P(t) = 3.93\,e^{0.02074t}$$

Malthusian Growth Model for U. S.

Modify Malthusian Growth Model Consider time-varying growth model

$$\frac{dP(t)}{dt} = k(t)\,P(t), \quad \text{with} \quad P(t_0) = P_0$$

- Assume $k(t) = a\,t + b$ is a linear function
- This is still a **linear differential equation**
- How do we solve this type of differential equation?
- What are the best constants a and b that fit the data for the U. S. population in 1790 and 1990?
- We must first learn about **Separable Differential Equations**

Separable Differential Equations 1

Separable Differential Equations Consider the differential equation

$$\frac{dy}{dt} = f(t, y)$$

- Assume the function $f(t, y)$ has the special separable form with

$$f(t, y) = M(t)N(y)$$

- Think of $\frac{dy}{dt}$ as the quotient of differentials
- We separate the differential equation in the following manner:

$$\begin{aligned} \frac{dy}{dt} &= M(t)N(y) \\ \frac{dy}{N(y)} &= M(t)dt \end{aligned}$$

Separable Differential Equations 2

Separable Differential Equations The differential equation

$$\frac{dy}{dt} = M(t)N(y)$$

- Separate so the left hand side has only the **dependent variable**, y, and the right hand side has only the **independent variable**, t
- The solution is obtained by integrating both sides

$$\int \frac{dy}{N(y)} = \int M(t)dt$$

Example 1 - Separable Differential Equation

Example - Separable Differential Equation Consider the differential equation

$$\frac{dy}{dt} = 2ty^2$$

Solution:

- Separate the variables t and y
  - Put only $2t$ and dt on the right hand side
  - And only y^2 and dy are on the left hand side
- The integral form is

$$\int \frac{dy}{y^2} = \int 2t\,dt$$

Example 1 - Separable Differential Equation

Solution (cont) The two integrals are

$$\int \frac{dy}{y^2} = \int 2t\,dt$$

- The two integrals are easily solved

$$-\frac{1}{y} = t^2 + C$$

- **Note** that you only need to put **one arbitrary constant**, despite solving two integrals
- This is easily rearranged to give the solution in explicit form

$$y(t) = -\frac{1}{t^2 + C}$$

Example 2 - Separable Differential Equation

Example 2: Consider the initial value problem

$$\frac{dy}{dt} = \frac{4\,\sin(2t)}{y} \quad \text{with} \quad y(0) = 1$$

Skip Example

Solution: Begin by separating the variables, so

$$\int y\,dy = 4\int \sin(2t)dt$$

Solving the integrals gives

$$\frac{y^2}{2} = -2\,\cos(2t) + C$$

Example 2 - Separable Differential Equation

Solution (cont) Since

$$\frac{y^2}{2} = -2\cos(2t) + C$$

We write

$$y^2(t) = 2C - 4\cos(2t) \qquad \text{or} \qquad y(t) = \pm\sqrt{2C - 4\cos(2t)}$$

From the initial condition

$$y(0) = 1 = \sqrt{2C - 4\cos(0)} = \sqrt{2C - 4}$$

Thus, $2C = 5$, and

$$y(t) = \sqrt{5 - 4\cos(2t)}$$

Example 3 - Separable Differential Equation

Example 3: Consider the initial value problem

$$\frac{dy}{dt} = -y\frac{(1+2\,t^2)}{t} \quad \text{with} \quad y(1) = 2$$

Skip Example

Solution: Begin by separating the variables, so

$$\int \frac{dy}{y} = -\int \frac{(1+2\,t^2)}{t}dt = -\int \frac{dt}{t} - 2\int t\,dt$$

Solving the integrals gives

$$\ln(y) = -\ln(t) - t^2 + C$$

Example 3 - Separable Differential Equation 2

Solution (cont): Since

$$\ln(y) = -\ln(t) - t^2 + C$$

Exponentiate both sides to give

$$y(t) = e^{-\ln(t)-t^2+C} = e^{-\ln(t)}e^{-t^2}e^{C} = \frac{A}{t}e^{-t^2}$$

where $A = e^C$
With the initial condition

$$y(1) = 2 = A\,e^{-1} \quad \text{or} \quad A = 2\,e^{1}$$

The solution is

$$y(t) = \frac{2}{t}e^{1-t^2}$$

Modified Malthusian Growth Model

Modified Malthusian Growth Model Consider the model

$$\frac{dP}{dt} = (a\,t + b)P \quad \text{with} \quad P(0) = P_0$$

- This equation is separable

$$\int \frac{dP}{P} = \int (a\,t + b)dt$$

- Integrating

$$\ln(P(t)) = \frac{a\,t^2}{2} + b\,t + C$$

- Exponentiating

$$P(t) = e^{\left(\frac{a\,t^2}{2} + b\,t + C\right)}$$

Modified Malthusian Growth Model

Modified Malthusian Growth Model: With $P(0) = e^C = P_0$, the model can be written

$$P(t) = P_0 e^{\left(\frac{a\,t^2}{2} + b\,t\right)}$$

- This model has 3 unknowns, P_0, a, and b
- As before, we fit the census data in 1790 aand 1990 of 3.93 million and 248.7 million
- Choose the third data value from the census in 1890, where the population is 62.95 million
- Again take t to be the years after 1790, then $P_0 = 3.93$

Modified Malthusian Growth Model

Population Model for U. S. The nonautonomous model is

$$P(t) = 3.93\, e^{\left(\frac{a\,t^2}{2} + b\,t\right)}$$

- Use the census data in 1890 and 1990 to find a and b
- The model gives

$$\begin{aligned} P(100) &= 62.95 = 3.93\, e^{5000a+100b} \\ P(200) &= 248.7 = 3.93\, e^{20000a+200b} \end{aligned}$$

- Taking logarithms, we have the linear equations

$$\begin{aligned} 5000\,a + 100\,b &= \ln\left(\frac{62.95}{3.93}\right) = 2.7737 \\ 20,000\,a + 200\,b &= \ln\left(\frac{248.7}{3.93}\right) = 4.1476 \end{aligned}$$

Modified Malthusian Growth Model

Population Model for U. S. Solving the linear equations

$$\begin{aligned} 5000\,a + 100\,b &= \ln\left(\frac{62.95}{3.93}\right) = 2.7737 \\ 20,000\,a + 200\,b &= \ln\left(\frac{248.7}{3.93}\right) = 4.1476 \end{aligned}$$

- Multiply the first equation by -2 and add to the second

$$10,000\,a = -2(2.7737) + 4.1476 = -1.3998$$

- Thus, $a = -0.00013998$, which is substituted into the first equation
- It follows that

$$100\,b = 5000(0.00013998) + 2.7737 = 3.473$$

- Solution is $a = -0.00013998$ and $b = 0.03473$

Modified Malthusian Growth Model

Population Models for U. S. The Malthusian growth model fitting the census data at 1790 and 1990 is

$$P(t) = 3.93\, e^{0.02074t}$$

The nonautonomous model fitting the census data at 1790, 1890, and 1990 is

$$P(t) = 3.93\, e^{0.03474t - 0.00006999t^2}$$

Model	1900	2000	2010
U. S. Census Data	76.21	281.4	308.7
Malthusian Growth	38.48	306.1	376.7
Nonautonomous	76.95	264.4	277.0

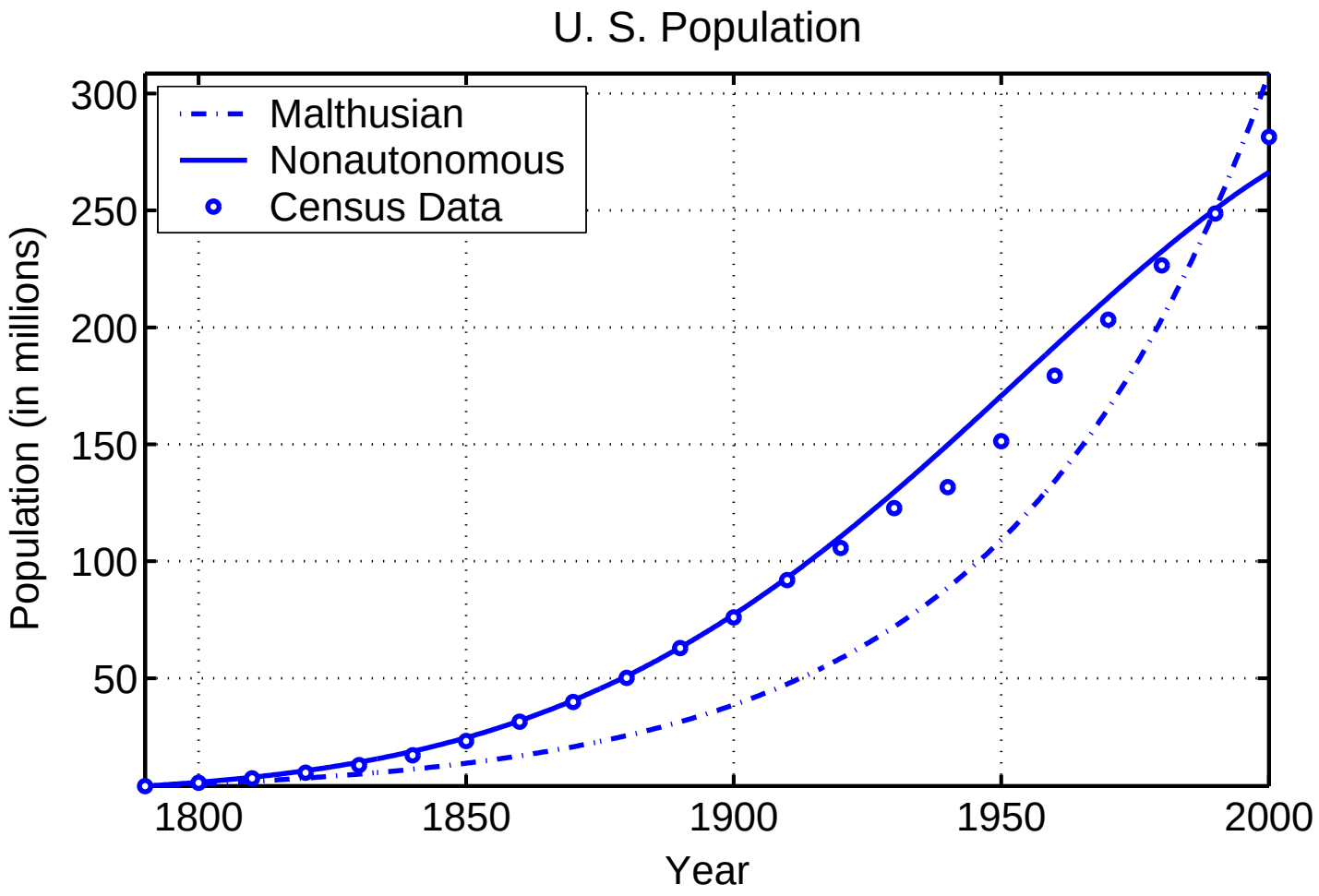
Modified Malthusian Growth Model

Population Models for U. S. The models use limited data for prediction

- For 1900
  - The **Malthusian growth model** is too low by 49.5%
  - The **nonautonomous growth model** is too high by 0.97%
  - The nonautonomous growth model fits quite well
- For 2000 and 2010
  - The **Malthusian growth model** is too high by 8.8% and 22%
  - The **nonautonomous growth model** is too low by 6.0% and 10.3%
  - Neither model fits the census data very well
  - The nonautonomous though fitting better misses the recent higher growth from immigration

Modified Malthusian Growth Model

Graphs of Population Models for U. S.



Population of Italy

Population of Italy: For the last few decades, Italy has had its growth rate decline to where the country does not even have enough births (or immigration) to replace the number of deaths in the country

- The population of Italy was 47.1 million in 1950, 53.7 million in 1970, and 56.8 million in 1990
- Use the data in 1950 and 1990 to find a Malthusian growth model for Italy's population
- Consider the nonautonomous Malthusian growth model given by the differential equation

$$\frac{dP}{dt} = (a\,t + b)P \quad \text{with} \quad P(0) = 47.1$$

with t in years after 1950

  - Solve this differential equation
  - Find the constants a and b from the data

Population of Italy

Population of Italy (cont):

- If the population of Italy was 50.2 million in 1960 and 57.6 million in 2000, then use each of these models to estimate the populations and determine the error between the models and the actual census values
- Graph the solutions of the two models and the data points from 1950 to 2000
- Find when Italy's population levels off and begins to decline according to the nonautonomous Malthusian growth model

Solution: The Malthusian growth model satisfies

$$\frac{dP}{dt} = rP \qquad \text{with} \qquad P(0) = 47.1$$

Population of Italy

Solution (cont): The solution of the Malthusian growth model is

$$P(t) = 47.1\, e^{rt}$$

- In 1990 the population was 56.8 million, so

$$P(40) = 47.1\, e^{40r} = 56.8$$

- Thus,

$$e^{40r} = \frac{56.8}{47.1} \quad \text{or} \quad r = \frac{1}{40} \ln\left(\frac{56.8}{47.1}\right) = 0.004682$$

- The Malthusian growth model for Italy is

$$P(t) = 47.1\, e^{0.004682\, t}$$

Population of Italy

Solution (cont): The nonautonomous Malthusian growth model is

$$\frac{dP}{dt} = (a\,t + b)P \qquad \text{with} \qquad P(0) = 47.1$$

- Separating variables

$$\int \frac{dP}{P} = \int (a\,t + b)dt$$

- Thus,

$$\ln(P(t)) = \frac{at^2}{2} + bt + c$$

- Exponentiating

$$P(t) = e^{\frac{at^2}{2}+bt+c} = e^c e^{\frac{at^2}{2}+bt}$$

Population of Italy

Solution (cont): The initial condition gives

$$P(0) = 47.1 = e^c$$

- The solution can be written

$$P(t) = 47.1\, e^{\frac{at^2}{2}+bt}$$

- The logarithmic form satisfies

$$\frac{at^2}{2} + bt = \ln(P(t)) - \ln(47.1)$$

- The data from 1970 and 1990 give

$$\begin{aligned} 200\,a + 20\,b &= \ln(53.7) - \ln(47.1) = 0.13114 \\ 800\,a + 40\,b &= \ln(56.8) - \ln(47.1) = 0.18726 \end{aligned}$$

Population of Italy

Solution (cont): The equations in a and b are linear equations

- Multiply the first equation by -2 and add it to the second

$$\begin{aligned} -2(200\,a + 20\,b) &= -2(0.13114) \\ 800\,a + 40\,b &= 0.18726 \\ 400\,a &= -0.07502 \end{aligned}$$

- It follows that $a = -0.00018755$
- From either equation above $b = 0.0084325$
- The solution becomes

$$P(t) = 47.1\,e^{0.0084325\,t - 0.00009378\,t^2}$$

Population of Italy

Solution (cont): The two models are given by

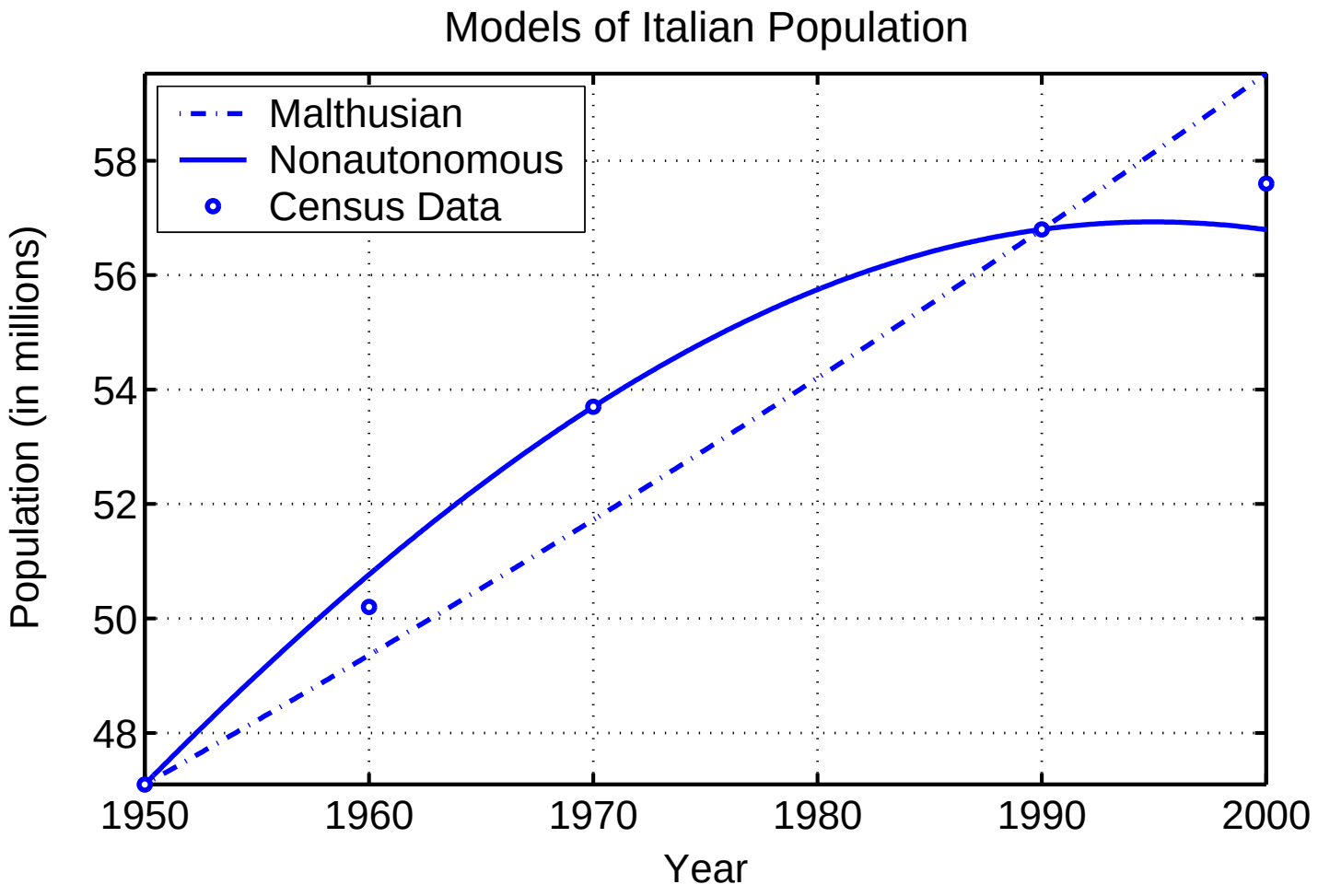
$$P(t) = 47.1\, e^{0.004682\, t} \quad \text{and} \quad P(t) = 47.1\, e^{0.0084325\, t - 0.00009378\, t^2}$$

Below is a Table comparing the models at 1960 and 2000

Model	1960	% Error	2000	% Error
Italy Census Data	50.2	–	57.6	–
Malthusian	49.4	−1.7%	59.5	3.3%
Nonautonomous	50.8	1.1%	56.8	−1.4%

Modified Malthusian Growth Model

Graphs of Population Models for Italy



Population of Italy

Solution (cont): The nonautonomous model is

$$\frac{dP}{dt} = (0.0084325 - 0.00018755\,t)P(t)$$

- The population growth slows to zero, so the population levels off, when $\frac{dP}{dt} = 0$
- This occurs when

$$0.0084325 - 0.00018755\,t = 0 \quad \text{or} \quad t = 44.96 \text{ years}$$

- The nonautonomous Malthusian growth model predicts that Italy's population leveled off in 1995 (45 years after 1950)
- Data indicates that 2000 was the peak of Italy's population, so the model does reasonably well

Desiccation of a Cell

Desiccation of a Cell: This example examines water loss through the surface of a cell

- Most cells are primarily water
- The loss of water due to desiccation is primarily through the surface of the cell
- Surface area varies proportionally to length squared, while volume varies according to length cubed
- The rate of change in the volume is proportional to the surface area to the 2/3 power
- An appropriate model for the desiccation of a cell is

$$\frac{dV}{dt} = -kV^{2/3}$$

where $V(t)$ is the volume of the cell

Desiccation of a Cell

Desiccation of a Cell: The model satisfies

$$\frac{dV}{dt} = -kV^{2/3}$$

- Suppose that the initial volume of water in the cell is $V(0) = 8 \text{ mm}^3$
- Suppose that 6 hours later the volume of water has decreased to $V(6) = 1 \text{ mm}^3$
- Solve this differential equation
- Find k and graph the solution
- Determine when all of the water has left the cell

Desiccation of a Cell

Solution: The model is a separable differential equation

$$\frac{dV}{dt} = -kV^{2/3}$$

- Separate variables to give

$$\int V^{-2/3} dV = -\int k\, dt$$

- Upon integration,

$$3V^{1/3}(t) = -kt + C$$

- Equivalently,

$$V(t) = \left(\frac{-kt + C}{3}\right)^3$$

- The initial condition gives

$$V(0) = 8 = \left(\tfrac{C}{3}\right)^3 \quad \text{or} \quad C = 6$$

Desiccation of a Cell

Solution: The model is given by

$$V(t) = \left(\frac{-kt+6}{3}\right)^3$$

- The other condition gives

$$V(6) = 1 = \left(\frac{-6k+6}{3}\right)^3 = (-2k+2)^3$$

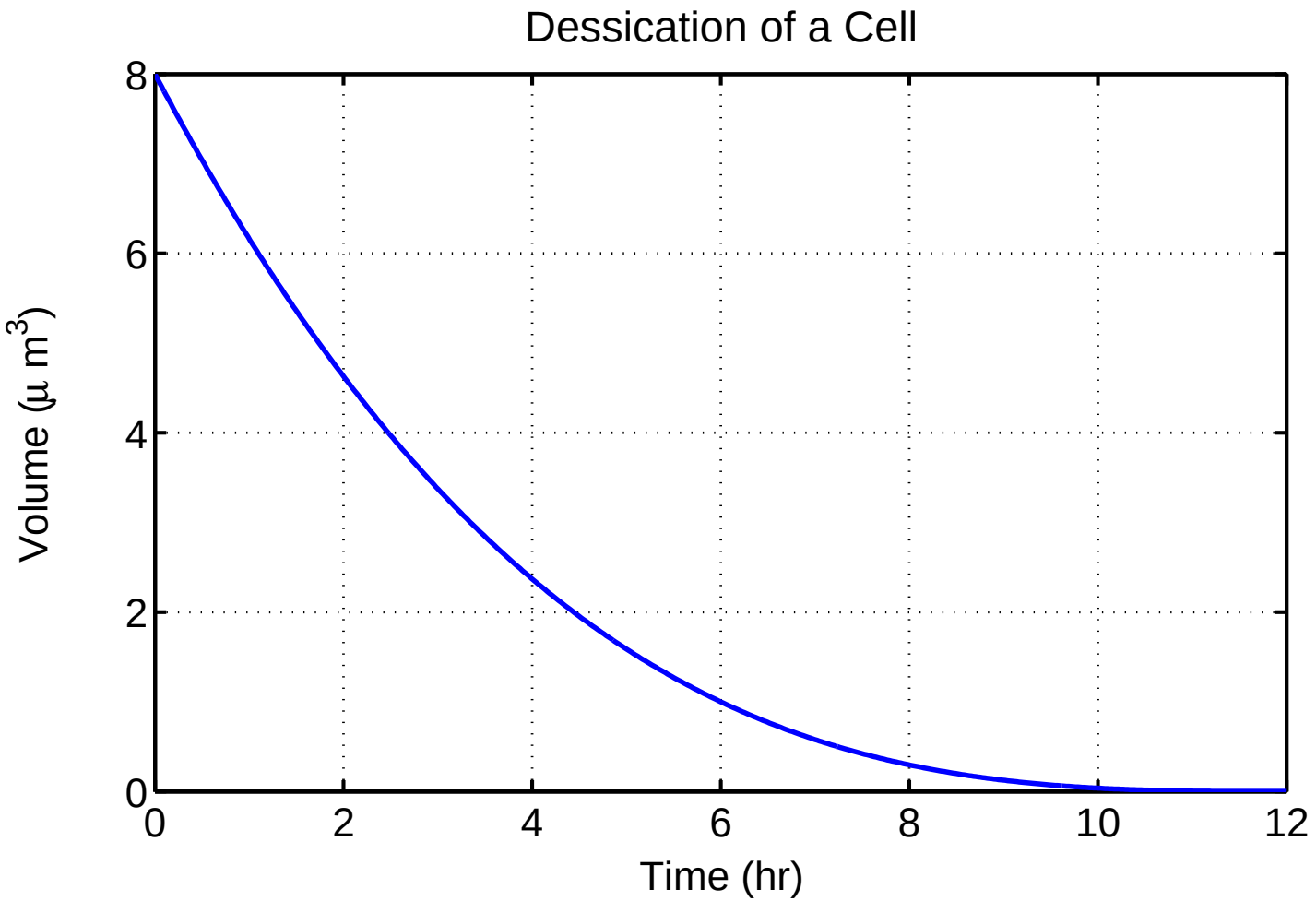
- So $k = \frac{1}{2}$
- The solution to this problem is

$$V(t) = \left(2 - \frac{t}{6}\right)^3$$

- The solution vanishes (all the water evaporates) at $t = 12$

Desiccation of a Cell

Graphs of Desiccation of a Cell



Water Height - Torricelli's Law

1

Water Height: Irrigation of vegetation from a leaking cylinder

- One method of delivering water at a slow rate for irrigation of vegetation is to put a small hole in the bottom of a cylindrical tank
- The water leaks out slowly over a period of time to provide extended irrigation
- Water flowing from a hole in the bottom of a reservoir of water satisfies Torricelli's law

Water Height - Torricelli's Law

2

Torricelli's Law: The rate of change of volume of water flowing from a reservoir (V) with a hole in the bottom of the tank is proportional to the square root of the height of the water above the hole (h)

- Mathematically, the law satisfies the differential equation:

$$\frac{dV}{dt} = -k\sqrt{h}$$

- This equation is derived using basic physics with the assumption that the sum of the kinetic and potential energy of the system remains constant

Water Height - Torricelli's Law

Modeling Water Height: The volume of water in the reservoir is equal to the cross-sectional area (A) of the cylinder times the height of the water (h) with A constant and $h(t)$ varying with time

$$V(t) = A\,h(t)$$

- It follows that

$$\frac{dV}{dt} = A\frac{dh}{dt}$$

- By Torricelli's Law, the model for the height is

$$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h}$$

Water Height - Torricelli's Law

4

Modeling Water Height: Suppose that a reservoir with a 20 cm radius begins with a height of 144 cm of water satisfies Torricelli's Law

$$\frac{dh}{dt} = -0.025\sqrt{h}$$

- Find the height of water in the reservoir at any time for this experimental irrigation system
- Determine how long until the reservoir is empty
- What is the average hourly amount of water (in cm^3/hr) delivered by this irrigation system
- Find the volume of water (in cm^3/hr) that is flowing after 100 hr and 800 hr

Water Height - Torricelli's Law

Solution: The model is

$$\frac{dh}{dt} = -0.025\, h^{1/2} \quad \text{with} \quad h(0) = 144$$

- This is a separable differential equation

$$\int h^{-1/2} dh = -\int 0.025\, dt$$

- Integrating gives

$$2\, h^{1/2} = -0.025\, t + C$$

- This equation is solved explicitly for $h(t)$

$$h(t) = \left(\frac{C}{2} - 0.0125\, t\right)^2$$

Water Height - Torricelli's Law

Solution (cont): The model is

$$h(t) = \left(\frac{C}{2} - 0.0125\,t\right)^2$$

- From the initial condition,

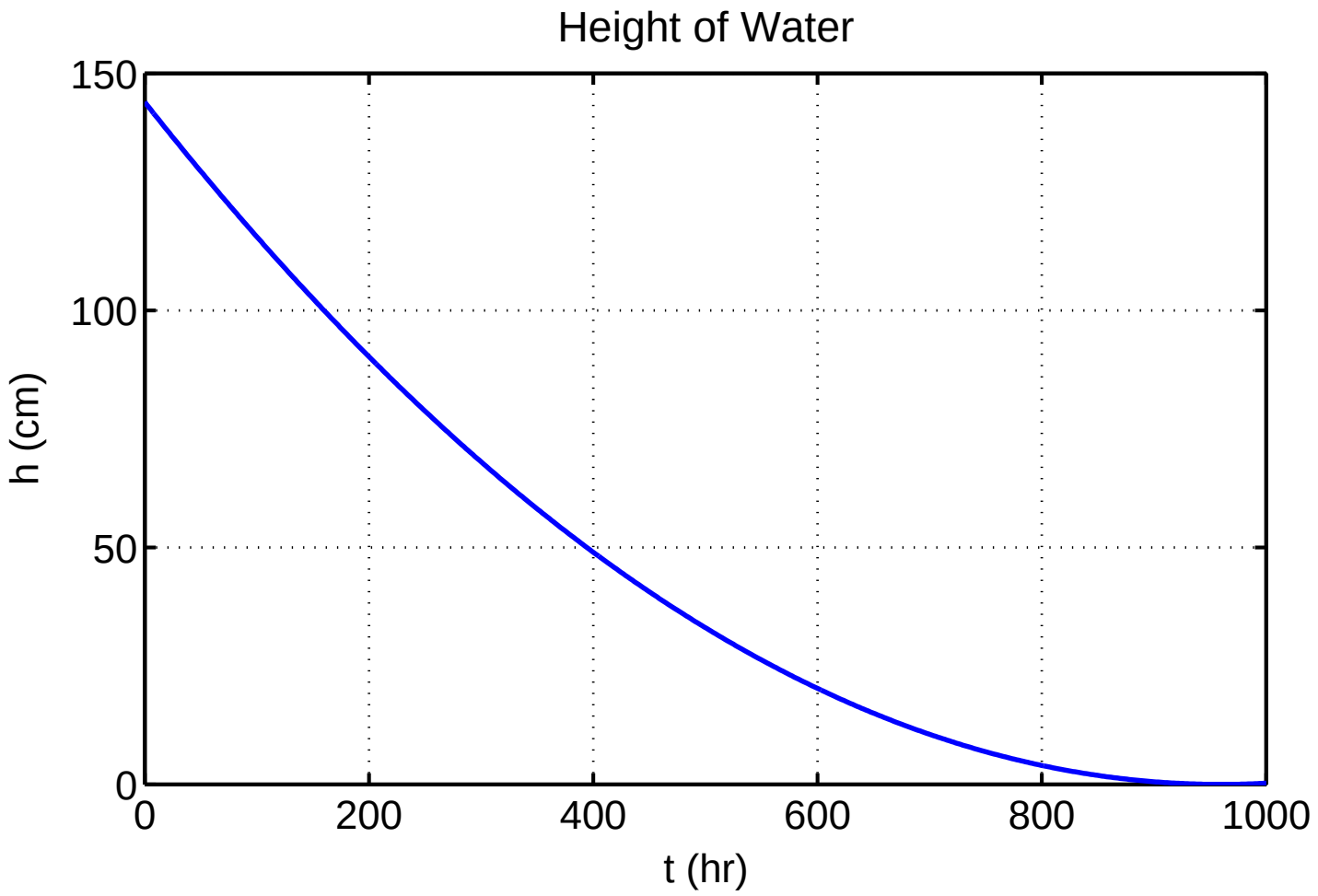
$$h(0) = 144 = \left(\frac{C}{2}\right)^2 \quad \text{or} \quad C = 24$$

- The solution is

$$h(t) = (12 - 0.0125\,t)^2$$

Water Height - Torricelli's Law

Graphs of Water Height



Water Height - Torricelli's Law

Solution (cont): The reservoir is empty when

$$h(t) = (12 - 0.0125\,t)^2 = 0$$

- Thus,

$$0.0125\,t = 12 \qquad \text{or} \qquad t = 960 \text{ hr}$$

- The reservoir empties in 960 hours or 40 days
- The total volume in the reservoir is

$$V = \pi(20)^2 144 = 57,600\pi = 180,956 \text{ cm}^3$$

- The average amount of water delivered over 960 hr is $\frac{180,956}{960} = 188.5 \text{ cm}^3/\text{hr}$

Water Height - Torricelli's Law

Solution (cont): The differential equation is used to find the water delivered at 100 and 800 hr

$$\frac{dV}{dt} = -0.025\, A\sqrt{h}$$

- The cross-sectional area satisfies

$$A = \pi(20)^2 = 400\pi = 1257 \text{ cm}^2$$

- The height of the water at $t = 100$ and 800 hr is

$$\begin{aligned} h(100) &= (12 - 0.0125(100))^2 &= 115.6 \text{ cm} \\ h(800) &= (12 - 0.0125(800))^2 &= 4.0 \text{ cm} \end{aligned}$$

Water Height - Torricelli's Law

Solution (cont): The volume of water flowing out is

$$\frac{dV}{dt} = -0.025(1257)\sqrt{h}$$

- The volume flowing out of the reservoir at $t = 100$ satisfies

$$\frac{dV}{dt} = -0.025(1257)\sqrt{115.6} = -337.7 \text{ cm}^3/\text{hr}$$

- This is above the average rate of water flowing out
- The volume flowing out of the reservoir at $t = 800$ satisfies

$$\frac{dV}{dt} = -0.025(1257)\sqrt{4.0} = -62.85 \text{ cm}^3/\text{hr}$$

- This is below the average rate of water flowing out