Calculus for the Life Sciences

Lecture Notes – Rules of Differentiation

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Outline

Introduction

- The previous section showed the definition of a derivative
- Using the definition of the derivative is not an efficient way to find derivatives
- Develop some rules for differentiation
- Basic power rule for differentiation
- Additive and scalar multiplication rules
- Applications to polynomials

Applications with Power Law

Pulse and Weight

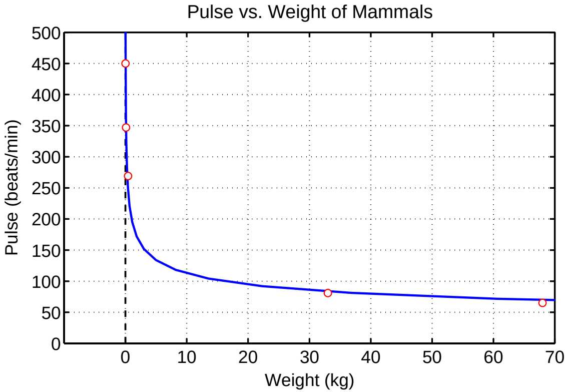
- Obtained data from Altman and Dittmer for the pulse and weight of mammals
- The pulse, P, as a function of the weight, w, are approximated by the relationship

$$P = 200w^{-1/4}$$

- The pulse is in beats/min, and the weight is in kilograms

Pulse and Weight

Pulse and Weight



Pulse and Weight

Pulse and Weight

- The graph shows an initial steep decrease in the pulse as weight increases
- Can one quantify how fast the pulse rate changes as a function of weight?
- For small animals the pulse rate changes more rapidly than for large animals
- The derivative of this allometric or power law model provides more details on the rate of change in pulse rate as a function of weight

Notation for the Derivative

Notation for the Derivative

- There are several standard notations for the derivative
- For the function $f(x)$, the notation that Leibnitz used was

$$\frac{df(x)}{dx}$$

- The Newtonian notation for the derivative is written as follows:

$$f'(x)$$

- We will use these notations interchangeably

Power Rule

Power Rule

The **power rule for differentiation** is given by the formula

$$\frac{d(x^n)}{dx} = nx^{n-1}, \qquad \text{for} \quad n \neq 0$$

Examples of the Power Rule

Examples: Differentiate the following functions:

If $f(x) = x^5$

The **derivative** is

$$f'(x) = 5\,x^4$$

If $f(x) = x^{-3}$

The **derivative** is

$$f'(x) = -3\,x^{-4}$$

If $f(x) = x^{1/3}$

The **derivative** is

$$f'(x) = \frac{1}{3}x^{-2/3}$$

Examples of the Power Rule

Examples: Differentiate the following functions:

If $f(x) = \frac{1}{x^4}$, then $f(x) = x^{-4}$

The **derivative** is

$$f'(x) = -4\,x^{-5}$$

If $f(x) = \frac{1}{\sqrt{x}}$, then $f(x) = x^{-1/2}$

The **derivative** is

$$f'(x) = -\frac{1}{2}\,x^{-3/2}$$

If $f(x) = 3$

Since $n = 0$, the power rule does not apply
However, we know the **derivative** of a **constant** is

$$f'(x) = 0$$

Scalar Multiplication Rule

Scalar Multiplication Rule

Assume that k is a constant and $f(x)$ is a differentiable function, then

$$\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx} f(x)$$

Example: Let $f(x) = 12\,x^3$

The derivative of $f(x)$ satisfies

$$f'(x) = \frac{d}{dx}(12\,x^3) = 12\frac{d}{dx}(x^3) = 36\,x^2$$

Additive Rule

Additive Rule

Assume that $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

Example: Let $f(x) = 2\,x^{1/2} + x^4$

The derivative of $f(x)$ satisfies

$$f'(x) = \frac{d}{dx}(2\,x^{1/2}) + \frac{d}{dx}(x^4) = x^{-1/2} + 4\,x^3$$

Differentiation of Polynomials

Differentiation of Polynomials

Consider the polynomial

$$f(x) = x^4 + 3\,x^3 - 8\,x^2 + 10\,x - 7$$

From our rules above, the derivative is

$$f\,'(x) = 4\,x^3 + 9\,x^2 - 16\,x + 10$$

Example: Other additive powers are handled similarly

$$f(x) = x^2 + \frac{3}{x^2} - 8\sqrt{x} + 13 = x^2 + 3\,x^{-2} - 8\,x^{1/2} + 13$$

From our rules above, the derivative is

$$f\,'(x) = 2\,x - 6\,x^{-3} - 4\,x^{-1/2}$$

Height of Ball

Height of Ball Suppose a ball is thrown vertically with an initial velocity of v_0 and an initial height $h(0) = 0$

Assume the only acceleration is due to gravity, g and air resistance ignored

The equation for the height satisfies:

$$h(t) = v_0 t - \frac{gt^2}{2}$$

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Velocity of Ball

With the equation for the height

$$h(t) = v_0 t - \frac{gt^2}{2}$$

The velocity is the derivative of $h(t)$

$$v(t) = h'(t) = v_0 - gt$$

- This uses our 3 rules of differentiation to date
  - The additive property of derivatives allows consideration of each of the terms in the height function separately
  - Each term has a scalar multiple
  - Power rule can be applied to the t and t^2 terms

Velocity of a Ball 1

Example: A ball, thrown vertically from a platform without air resistance, satisfies the equation

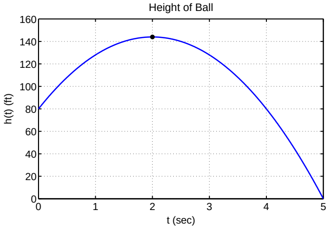
$$h(t) = 80 + 64\,t - 16\,t^2$$

- Sketch a graph of the height of the ball, $h(t)$, as a function of time, t
- Find the maximum height of the ball and determine when the ball hits the ground
- Give an expression for the velocity, $v(t)$, as a function of time, t
- Find the velocity at the times $t = 0$, $t = 1$, and $t = 2$
- What is the velocity of the ball just before it hits the ground?

Velocity of a Ball 2

Height of the ball

- Factoring $h(t) = -16(t+1)(t-5)$, so the ball hits the ground at $t = 5$
- The vertex of the parabola occurs at $t = 2$ with $h(2) = 144$ ft
- The h-intercept is $h(0) = 80$ ft



Velocity of a Ball

Since the height is given by

$$h(t) = 80 + 64\,t - 16\,t^2$$

so the velocity is

$$v(t) = h'(t) = 64 - 32\,t$$

- It follows that

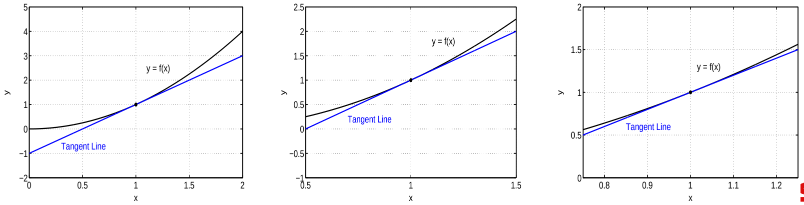
$$v(0) = 64 \text{ ft/sec}, \quad v(1) = 32 \text{ ft/sec}, \quad v(2) = 0 \text{ ft/sec}$$

- The velocity at the maximum is $v(2) = 0$ ft/sec
- The ball hits the ground with velocity $v(5) = -96$ ft/sec

Linear Approximation

Linear Approximation

- Recall that the **tangent line** gives a **linear approximation** of a function near the point of tangency
  - The **derivative** give the slope of this tangent line
  - A point on the curve gives the point of tangency
- This provides easy approximations of a function near a given point
- This technique is often used in **Error Analysis**



Pulse and Weight Example 1

Pulse and Weight Example

The model on pulse rate is,

$$P = 200\, w^{-0.25}$$

The power law of differentiation gives

$$\frac{dP}{dt} = -50\, w^{-5/4}$$

The negative sign shows the decrease in the pulse rate with increasing weight

Pulse and Weight Example 2

Example for Linear Approximation: Suppose we want to approximate the pulse of a 17 kg animal using our model

$$P = 200\,w^{-0.25}$$

- An animal at 16 kg by the allometric model would have a pulse of about 100 (since $P(16) = 200(16)^{-1/4} = 100$)
- The power law of differentiation gives

$$\frac{dP}{dt} = -50\,w^{-5/4}$$

- The derivative at $w = 16$ is

$$P'(16) = -50(16)^{-5/4} = -\frac{50}{32} \approx -1.56$$

Pulse and Weight Example

Example for Linear Approximation (cont):

- The tangent line approximation, $P_L(w)$, near $w = 16$ is

$$P_L(w) = -\frac{50}{32}(w - 16) + 100$$

- It follows that a 17 kg animal should have a pulse near

$$P_L(17) = -\frac{50}{32}(1) + 100 \approx 98.44 \text{ beats/min}$$

- Note that the **Allometric model** gives

$$P(17) = 200(17)^{-1/4} = 98.50 \text{ beats/min}$$

Biodiversity and Area

Biodiversity and Area

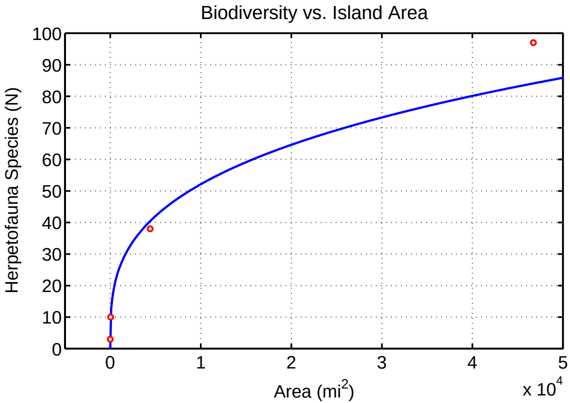
- Data are collected on the number of species of herpatofauna, N, on Caribbean islands with area, A
- An allometric model approximates this biodiversity

$$N = 3A^{1/3}$$

- A model of this sort is important for obtaining information about biodiversity

Biodiversity and Area

Biodiversity and Area



Biodiversity and Area

Biodiversity and Area

- Can we use this model to determine the rate of change of numbers of species with respect to a given increase in area?
- Again the derivative is used to help quantify the rate of change of the dependent variable, N, with respect to the independent variable, A

Biodiversity Example 1

Biodiversity Example

The model on diversity is,

$$N = 3\,A^{1/3}$$

The power law of differentiation gives

$$\frac{dN}{dt} = A^{-2/3}$$

- This shows the rate of change of numbers of species with respect to the island area is increasing as the derivative is positive
- The increase gets smaller with increasing island area, since the area has the power $-2/3$, which puts the area in the denominator of this expression for the derivative

Skip Linear Approximation

Biodiversity Example 2

Example for Linear Approximation: Suppose we want to approximate the number of species on an island with 950 sq mi

$$N = 3\,A^{1/3}$$

- An island with 1000 sq mi by the allometric model would have approximately 30 species (since $N(1000) = 3(1000)^{1/3} = 30$)
- The power law of differentiation gives

$$\frac{dN}{dt} = A^{-2/3}$$

- The derivative at $A = 1000$ is

$$N'(1000) = (1000)^{-2/3} = 0.01$$

Biodiversity Example 3

Example for Linear Approximation (cont):

- The tangent line approximation, $N_L(A)$, near $A = 1000$ is

$$N_L(A) = 0.01(A - 1000) + 30$$

- It follows that an island with an area of 950 sq mi should have approximately

$$N_L(950) = 0.01(950 - 1000) + 30 = 29.5 \text{ species}$$

- Note that the **Allometric model** gives

$$N(950) = 3(950)^{1/3} = 29.49 \text{ species}$$

Logistic Growth Model 1

A common model in population biology is the **logistic growth model** given by

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{M}\right)$$

- Studied the **discrete Malthusian growth model**
  - The growth of the population is proportional to the existing population, $P_{n+1} = P_n + rP_n$
  - Malthusian growth model is based on unlimited resources
- As the population increases, the growth rate of most organisms slows
  - Crowding (lack of space to reproduce)
  - Lack of resources (limited food supply)
  - Build up of waste (toxicity)

Logistic Growth Model 2

The **logistic growth model**

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{M}\right) = P_n + G(P_n)$$

- First part is same as Malthusian growth model
- Quadratic term reflects slowing of growth with increasing population, **growth function**, $G(P_n)$
- Nonlinear model, which can have complicated behavior (observe later in Lab)
- For low r values, model gives classic **S-shaped curve**
- Population reaches an equilibrium, the **carrying capacity**

Logistic Growth Model

Example of Logistic Growth Function: Suppose a culture of yeast has the growth function

$$G(P) = rP\left(1 - \frac{P}{M}\right)$$

where P is the density of yeast $(\times 1000/\text{cc})$

- Suppose experimental measurements find the growth parameters
  - The Malthusian growth rate $r = 0.1$
  - The parameter $M = 500$

Logistic Growth Model 4

The population is at **equilibrium** when the growth function is zero

$$G(P) = 0.1\,P\left(1 - \frac{P}{500}\right) = 0$$

- This quadratic growth function is in factored form, so equilibria are easily found
  - The **extinction** equilibrium, $P_e = 0$
  - The **carrying capacity**, $P_e = M = 500$

Logistic Growth Model

The **maximum growth** occurs at the vertex of the growth function

Also, the **maximum** is when the slope of the tangent line is **zero** or the **derivative** is **zero**

Since

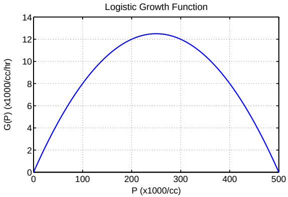
$$G(P) = 0.1\,P - \frac{0.1\,P^2}{500}$$

the derivative is

$$G'(P) = 0.1 - \frac{0.2\,P}{500}$$

$G'(P) = 0$ when $P = 250$

Logistic Growth Model



- This model gives **equilibria** at $P_e = 0$ and $P_e = 500$ ($\times 1000$) yeast/cc
- Maximum population growth occurs at $P_v = 250$ ($\times 1000$) yeast/cc
- Since $G(250) = 12.5$, when the **density of yeast** is 250 ($\times 1000$) yeast/cc, the maximum production is 12.5 ($\times 1000$) yeast/cc/hr

Logistic Growth Model

- Suppose the population begins with $P_0 = 50$ ($\times 1000$) yeast/cc
- Below shows the simulation of

$$P_{n+1} = P_n + 0.1\, P_n \left(1 - \frac{P_n}{500}\right)$$

for $0 \leq n \leq 80$ hr

- Simulation shows the population approaching the **carrying capacity** of 500 and the maximum growth near $n = 25$