Calculus for the Life Sciences

Lecture Notes – Definite Integral

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Outline

Introduction

Introduction

- Riemann Integral and Numerical Methods of Integration approximated the area under a curve
- Midpoint Rule used a large number of rectangles
- This section connects integrals using antiderivatives to area under a curve
- The **Fundamental Theorem of Calculus** allows the use of the definite integral to find the exact area under a function

Respiratory Dead Space

Respiratory Dead Space

- When breathing air in and out of the lungs, the air must pass through the nasal passageways, the pharnyx, the trachea, and the bronchi before it can enter the alveoli where the oxygen and carbon dioxide exchange with the circulatory system
- These regions where vital gases cannot be exchanged are called **dead spaces**
- To determine the health of patients with respiratory problems, it is important to know information on all aspects of their lungs
- This includes the measurement of the dead space

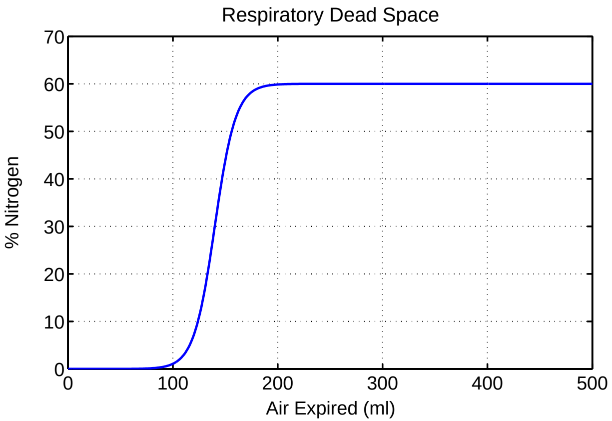
Respiratory Dead Space 2

Respiratory Dead Space is simple to measure

- The patient breathes normal air, then takes a single breath of pure oxygen
- The oxygen mixes with the normal air in the alveoli
- The dead space is filled almost exclusively with pure oxygen
- The patient expires the mixture through a rapidly recording nitrogen meter
- The recording gives a measurement of the amount of N_2, and the part that includes only O_2 represents the dead space

Respiratory Dead Space

Graph of Respiratory Dead Space



Respiratory Dead Space 4

Respiratory Dead Space

- The region to the left of the curve is the pure O_2 in the dead space
- The region to the right of the curve represents the mixed air in the alevoli where that actual gas is being exchanged with the circulatory system
- The volume of the dead space is given by the area to the left of the curve times the total volume of air expired divided by the total area under the 60% level

Respiratory Dead Space

Respiratory Dead Space function fit to the data

$$N(x) = 0.3 + 0.3\frac{e^{0.05(x-140)} - e^{-0.05(x-140)}}{e^{0.05(x-140)} + e^{-0.05(x-140)}}$$

- N is the percent of nitrogen in the expired air and x is the number of ml expired
- The total area (V) under the 60% line is

$$V = 0.6 \times 500 = 300 \text{ ml}$$

- We find the area to the left of the curve by finding the area under the curve and subtracting it from the total area, V
- This area is found using **Fundamental Theorem of Calculus**

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

- Let $f(x)$ be a continuous function on the interval $[a, b]$ and assume that $F(x)$ is any antiderivative of $f(x)$
- The **definite integral**, which gives the area under the curve $f(x)$ between a and b, is computed by the following formula:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Properties of Definite Integral

Properties of Definite Integral: Assume $f(x)$ be a continuous function on the interval $[a, b]$

1. **Reversing Endpoints**:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2. **Odd function**: Assume a symmetric interval $x \in [-a, a]$:

$$\int_{-a}^a f(x)dx = 0$$

3. **Even function**: Assume a symmetric interval $x \in [-a, a]$:

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

Example 1

Example 1: Use the Fundamental Theorem of Calculus to evaluate the integral of

$$f(x) = x^2 \qquad x \in [0, 2]$$

Skip Example

Solution: The solution is given by

$$\begin{aligned} \int_0^2 x^2 dx &= \left. \frac{x^3}{3} \right|_0^2 \\ &= \frac{8}{3} - \frac{0}{3} = \frac{8}{3} \end{aligned}$$

This represents the area under the curve x^2 from 0 to 2

Example 2

Example 2: Consider the functions

$$f(x) = x^2 - 2x - 3 \qquad \text{and} \qquad g(x) = 1 - 2x$$

Skip Example

- Find the x and y-intercepts and the vertex of the parabola
- Find the points of intersection
- Determine the area between the curves

Example 2

Solution: For $f(x) = x^2 - 2x - 3 = (x-3)(x+1)$

- The y-intercept is $(0, -3)$
- The x-intercepts are $(-1, 0)$ and $(3, 0)$
- Since the midpoint between the x-intercepts is $x = 1$, so the vertex is $(1, -4)$
- For the line $g(x) = 1 - 2x$, the x and y-intercepts are $\left(\frac{1}{2}, 0\right)$ and $(0, 1)$

Example 2

3

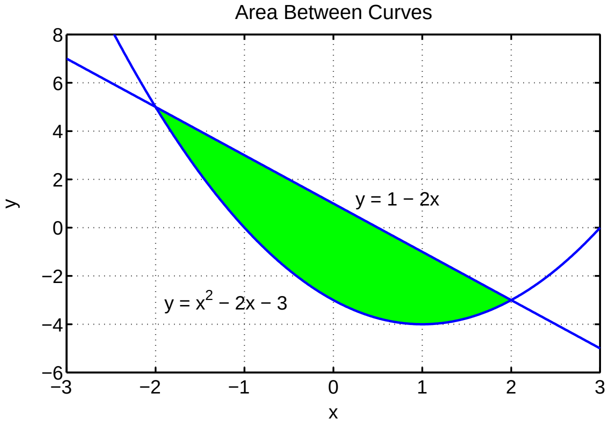
Solution: Points of intersection are found by solving

$$x^2 - 2x - 3 = 1 - 2x$$

- Thus, $x^2 - 4 = 0$
- The x values of intersection are $x = \pm 2$
- The points of intersection are $(-2, 5)$ and $(2, -3)$

Example 2

Graph of the Two Curves



Example 2

Area Between the Curves: Notice that $g(x) \geq f(x)$ for $x \in [-2, 2]$

The height at any x is

$$g(x) - f(x) = (1 - 2x) - (x^2 - 2x - 3) = 4 - x^2$$

$$\begin{aligned} \int_{-2}^{2} (4 - x^2)dx &= \left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2} \\ &= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) \\ &= \frac{32}{3} \end{aligned}$$

Example 3

Example 3: Evaluate the following definite integral:

$$\int_0^4 2\sqrt{2t+1}\,dt$$

Skip Example

Example 3

Solution: For this integral we need the substitution

$$u = 2t + 1, \qquad \text{so} \qquad du = 2\,dt$$

The endpoints are $t = 0$, which changes to $u = 1$, and $t = 4$, which becomes $u = 9$

$$\begin{aligned} \int_0^4 \sqrt{2t+1}\,(2)\,dt &= \int_1^9 u^{1/2}du \\ &= \frac{2}{3}u^{3/2}\Big|_1^9 \\ &= \frac{2}{3}\left(9^{3/2} - 1\right) = \frac{52}{3} \end{aligned}$$

Volume of the Dead Space 1

Solution: The dead space for breathing is found by determining the area of the region to the left of the curve The area of that region can be approximated by the definite integral

$$\int_0^{500}(0.6-N(x))dx = \int_0^{500}\left(0.3 - 0.3\frac{e^{0.05(x-140)} - e^{-0.05(x-140)}}{e^{0.05(x-140)} + e^{-0.05(x-140)}}\right)dx$$

Break the integral into two integrals

$$\int_0^{500} 0.3\,dx = 0.3x\Big|_0^{500} = 150$$

Volume of the Dead Space 2

Solution: The second integral

$$0.3\int_0^{500}\left(\frac{e^{0.05(x-140)}-e^{-0.05(x-140)}}{e^{0.05(x-140)}+e^{-0.05(x-140)}}\right)dx$$

Make the substitution

$$u = e^{0.05(x-140)} + e^{-0.05(x-140)} \quad \text{with}$$

$$du = 0.05\left(e^{0.05(x-140)} - e^{-0.05(x-140)}\right)dx$$

The endpoints change from $x = 0$ to

$$u = e^{-7} + e^{7} \approx e^{7}$$

and for $x = 500$ to

$$u = e^{18} + e^{-18} \approx e^{18}$$

Volume of the Dead Space 3

Solution: With the substitutions

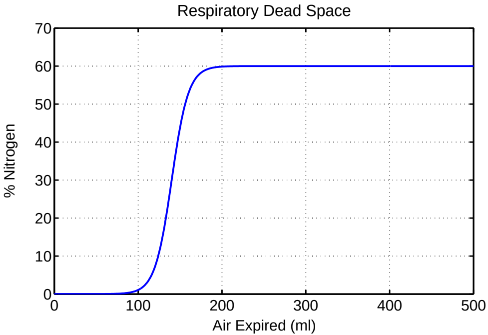
$$\begin{aligned} 6\int_0^{500}\left(\frac{e^{0.05(x-140)}-e^{-0.05(x-140)}}{e^{0.05(x-140)}+e^{-0.05(x-140)}}\right)(0.05)dx &= 6\int_{e^7}^{e^{18}}\frac{du}{u} \\ &= 6\ln(u)\Big|_{e^7}^{e^{18}} \\ &= 6(18-7) \\ &= 66 \end{aligned}$$

Combining the above results

$$\int_0^{500}\left(0.3-0.3\frac{e^{0.05(x-140)}-e^{-0.05(x-140)}}{e^{0.05(x-140)}+e^{-0.05(x-140)}}\right)dx = 150-66=84$$

Volume of the Dead Space

- The measured volume of nitrogen N_2 lost in the deadspace to pure oxygen O_2 is 84 ml
- The limiting nitrogen in the graph below is only 60%, so the actual deadspace is $84/0.6 = 140$ ml



Example 4

Example 4: Evaluate the following definite integral:

$$\int_{-1}^{1} x^3 dx$$

Skip Example

$$\int_{-1}^{1} x^3 dx = \left.\frac{x^4}{4}\right|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

- This definite integral has no net area under the curve
- A function with odd symmetry over an interval centered on the origin results in the integral being zero (See Properties)

Example 5

Example 5: Evaluate the following definite integral:

$$\int_0^{\pi/2} (2 - \sin(t))^2 \cos(t) dt$$

Skip Example

Make the substitution $u = 2 - \sin(t)$ with $du = -\cos(t)dt$

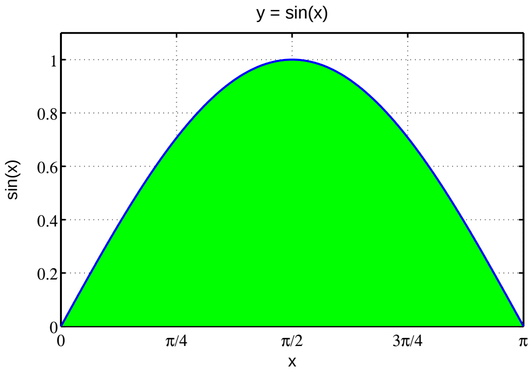
The endpoints give $t = 0$, which changes to $u = 2$
and $t = \frac{\pi}{2}$, which changes to $u = 1$

$$\begin{aligned} -\int_0^{\pi/2} (2 - \sin(t))^2 (-\cos(t)) dt &= -\int_2^1 u^2 du = \int_1^2 u^2 du \\ &= \left. \frac{u^3}{3} \right|_1^2 \\ &= \tfrac{1}{3}(8 - 1) = \tfrac{7}{3} \end{aligned}$$

Example 6: Area 1

Example 6: Find the area under the sine curve for $x \in [0, \pi]$

Skip Example



Example 6: Area

Solution: The area is found by integrating $\sin(x)$ on the interval from 0 to π

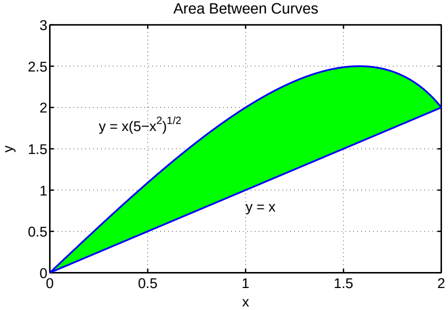
$$\begin{aligned}\int_0^{\pi} \sin(x)dx &= -\cos(x)\Big|_0^{\pi} \\ &= -\cos(\pi)+\cos(0) \\ &= 2\end{aligned}$$

Example 7: Area

Example 7: Find the area between the curves $f(x)$ and $g(x)$, where

$$f(x) = x\sqrt{5-x^2} \quad \text{and} \quad g(x) = x$$

Skip Example



Example 7: Area

Solution: The points of intersection are found when $f(x) = g(x)$

$$x = x\sqrt{5 - x^2} \qquad \text{or} \qquad x = 0 \quad \text{and} \quad \sqrt{5 - x^2} = 1$$

The points of intersection occur at $x = 0$ and 2

The area between the curves satisfies the definite integral

$$\int_0^2 \left(x\sqrt{5 - x^2} - x \right) dx = \int_0^2 x\sqrt{5 - x^2}dx - \int_0^2 x\,dx$$

Example 7: Area

Solution: For the integral

$$\int_0^2 x\sqrt{5-x^2}dx - \int_0^2 x\,dx$$

The first integral requires the substitution

$$u = 5 - x^2, \qquad \text{so} \qquad du = -2x\,dx$$

The substitution changes the limits of integration from $x = 0$ to $u = 5$
and $x = 2$ to $u = 1$

$$-\frac{1}{2}\int_0^2 \sqrt{5-x^2}(-2x)dx - \int_0^2 x\,dx = -\frac{1}{2}\int_5^1 u^{1/2}du - \int_0^2 x\,dx$$

Example 7: Area 4

Solution: The substituted integral is much easier to solve

$$\begin{aligned}
\frac{1}{2}\int_1^5 u^{1/2}du - \int_0^2 x\,dx &= \left.\frac{u^{3/2}}{3}\right|_1^5 - \left.\frac{x^2}{2}\right|_0^2 \\
&= \tfrac{1}{3}(5\sqrt{5}-1) - \tfrac{1}{2}(4-0) \\
&= -\tfrac{7}{3} + \tfrac{5}{3}\sqrt{5} \approx 1.39
\end{aligned}$$

Thus, the area between the two curves is approximately 1.39

Example 8: Average Population 1

Example 8: Average Population A sample plot of grassland is surveyed for a particular species of insect

Week	0	1	2	5	9	10	12
Population	403	255	176	230	478	504	398

A reasonable fit to the data is given by the cubic polynomial

$$f(t) = 400 - 180t + 39t^2 - 2t^3$$

Skip Example

- Use the polynomial approximation to predict the maximum and minimum populations
- Graph the data and the polynomial
- Use the data to find the average number of insects in the plot, then use the approximating polynomial to estimate the average number of insects in the plot

Example 8: Average Population 2

Solution: The polynomial fit to the data is

$$f(t) = 400 - 180t + 39t^2 - 2t^3$$

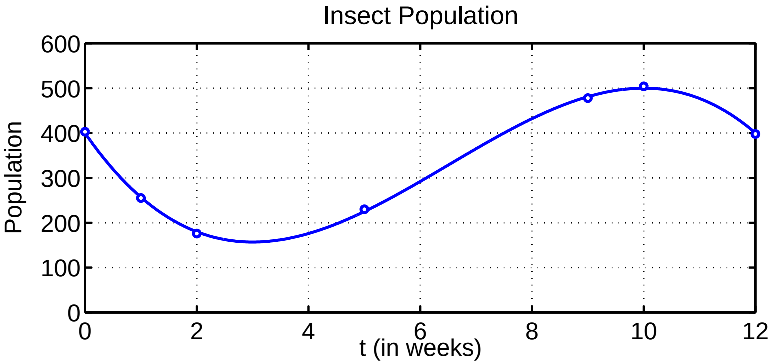
The derivative is

$$f'(t) = -180 + 78t - 6t^2 = -6(t-3)(t-10)$$

- The relative extrema occur at $t = 3$ with $f(3) = 157$ and $t = 10$ with $f(10) = 500$
- The endpoints are $f(0) = 400$ and $f(12) = 400$
- The polynomial predicts that the minimum population occurs at $t = 3$ with a population of 157
- The maximum population occurs at $t = 10$ with a population of 500

Example 8: Average Population 3

Graph of polynomial and data



Example 8: Average Population 4

Average: The average of the data given above is easily seen to be 349.1

To find the average population using the definite integral

$$\begin{aligned} P_{ave} &= \frac{1}{12}\int_0^{12}(400 - 180\,t + 39\,t^2 - 2\,t^3)dt \\ &= \frac{1}{12}\left(400\,t - 90\,t^2 + 13\,t^3 - \tfrac{1}{2}\,t^4\right)\Big|_0^{12} \\ &= 400 - 90(12) + 13(12)^2 - \tfrac{1}{2}(12)^3 - 0 \\ &= 328 \end{aligned}$$

Example 8: Average Population 5

Averages:

- There is a slight difference between the averages
- The better average depends on how you want to interpret your data
- In general, the average given by the integral is more representative because it gives an even weighting over the time of the experiment

Example 9: Radiation Exposure 1

Example 9: Radiation Exposure: The integral can be used to calculate the dose from radiation exposure

- If the substance has a very long half-life, then the radiation exposure is easily approximated by simply multiplying the amount of radiation measured times the length of time of exposure
- If the radioactive source is decaying rapidly, then the radiation exposure varies with time

Example 9: Radiation Exposure 2

Example 9: Radiation Exposure: Phosphorous ^{32}P has a half-life of 14.3 days, undergoing an energetic beta decay (1.7MeV)

- This radioactive tracer is often used in biological experiments to label nucleotides
- Suppose that a hot sample of ATP emits 300 mREM/day
- Solve the differential equation

$$\frac{dR}{dt} = -kR, \qquad R(0) = 300$$

- The Nuclear Regulatory Commission (NRC) specifies that a laboratory worker can receive 5000 mREM/yr
- If the lab worker stays close to this sample, then determine how long the lab worker could remain near the sample before receiving his or her total annual radiation dose

Example 9: Radiation Exposure

Solution: The differential equation

$$\frac{dR}{dt} = -kR, \qquad R(0) = 300$$

has the solution

$$R(t) = 300\, e^{-kt}$$

Because the half-life is 14.3 days,

$$150 = 300\, e^{-14.3k} \qquad \text{or} \qquad e^{14.3k} = 2$$

Thus,

$$k = \frac{\ln(2)}{14.3} = 0.0485$$

The solution is given by

$$R(t) = 300e^{-0.0485t}$$

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Example 9: Radiation Exposure

Solution: The dose that a lab worker receives is based on the rate of exposure times the length of time that the worker is exposed

For rapidly decaying radioactive materials, this is given by an integral of the radioactive decay function integrated over the time of exposure

For the sample of ^{32}P given, the exposure is

$$\int_0^t R(s)ds = 300\int_0^t e^{-0.0485s}ds$$

where t is the time the lab worker is exposed

The allowable dose satisfies the equation

$$300\int_0^t e^{-0.0485s}ds = 5000$$

Example 9: Radiation Exposure 5

Solution: Need to solve the integral equation for t

$$300\int_0^t e^{-0.0485s}ds = 5000$$

The solution gives

$$\begin{aligned}\int_0^t e^{-0.0485s}ds &= \frac{50}{3}\\ \left.\frac{e^{-0.0485t}}{-0.0485}\right|_0^t &= \frac{50}{3}\\ e^{-0.0485t}-1 &= -0.808\\ e^{-0.0485t} &= 0.1917\\ -0.0485\,t &= \ln(0.1917)\end{aligned}$$

The allowable dose of 5000 mREM/yr in $t = 34.0$ days

Example 9: Radiation Exposure 6

Solution: This lab sample gives the allowable annual body dose in $t = 34.0$ days

- Note that the lab worker is not likely to be around the sample all the time
- Radiation satisfies an inverse square law, so moving some distance away from the sample dramatically lowers exposure
- This sample is unlikely to cause significant exposure even though it is a fairly hot sample