

# Calculus for the Life Sciences

## Lecture Notes – Chain Rule

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# Outline

- 1 Height and Weight
  - Introduction
  - Average Height and Weight of Girls
- 2 Chain Rule
  - Examples
  - Chain Rule for Special Functions
  - More Examples
- 3 Applications
  - Rate of Change in Weight
  - Normal Distribution
  - Hassell's Model

# Chain Rule

## Chain Rule

- Functional relationships where one measurable quantity depends on another, while the second quantity is a function of a third quantity
- This functional relationship is a **composite function**
- The differentiation of a composite function requires the **chain rule**

# Average Height and Weight of Girls

1

## Average Height and Weight of Girls

- Over a range of ages the rate of growth of girls in height is constant
- Height and age are approximated well by a **linear function**
- Height and weight of animals satisfies an **allometric model**

## Average Height and Weight of Girls

2

## Average Height and Weight of American Girls

age (years)	height (cm)	weight (kg)	age (years)	height (cm)	weight (kg)
1	75	9.5	8	126	25.0
2	87	11.8	9	132	29.1
3	94	15.0	10	138	32.7
4	102	15.9	11	144	37.3
5	108	18.2	12	151	41.4
6	114	20.0	13	156	46.8
7	121	21.8			

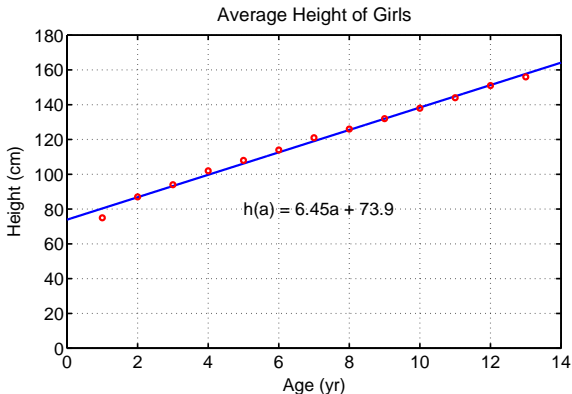
## Average Height and Weight of Girls

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**Least Squares Best Fit:** Model of Height as a function of age

$$h(a) = 6.45a + 73.9$$

Model shows that the average girl grows about **6.45 cm/yr**

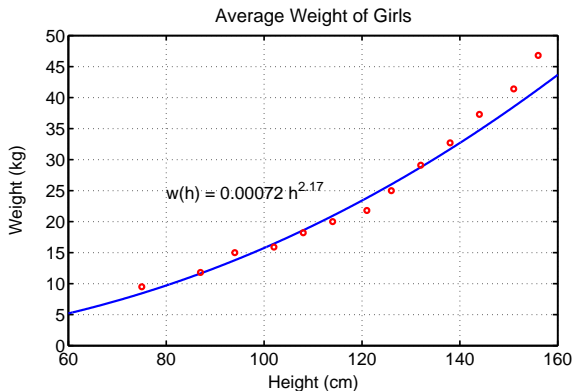


## Average Height and Weight of Girls

4

**Allometric Model:** An allometric model for the height and weight of a girl satisfies

$$W(h) = 0.000720 h^{2.17}$$



## Average Height and Weight of Girls

**Composite Model:** The **linear model** shows that the average girl grows about **6.45 cm/yr**

- How do we find the **rate of change in weight** for a girl at any particular age (between 1 and 13)?
  - The **Allometric Model** gives the weight as a function of height
  - Create a **composite function** of the **allometric model** and the **linear model** to give a function of the weight as a function of age
  - The **chain rule** gives the rate of change of weight with respect to age



# Chain Rule

**Chain Rule:** Consider the **composite function**  $f(g(x))$

- Suppose that both  $f(u)$  and  $u = g(x)$  are differentiable functions
- The **chain rule** for differentiation of this composite function is given by

$$\frac{df}{dx} = \frac{df}{du} \frac{du}{dx}$$

- Alternately, the chain rule is written

$$\frac{d}{dx} (f(g(x))) = f'(g(x))g'(x)$$

## Example – Chain Rule

**Example - Chain Rule:** Consider the the function

$$h(x) = (x^2 + 2x - 5)^5$$

Find  $h'(x)$

**Solution:** Consider the composite of the functions

$$f(u) = u^5 \quad \text{and} \quad g(x) = x^2 + 2x - 5$$

The derivatives of both  $f$  and  $g$  are

$$f'(u) = 5u^4 \quad \text{and} \quad g'(x) = 2x + 2$$

From the **chain rule**

$$h'(x) = 5(g(x))^4(2x + 2)$$

$$h'(x) = 5(x^2 + 2x - 5)^4(2x + 2)$$

## Example 2 – Chain Rule

**Example 2 - Chain Rule:** Consider the the function

$$h(x) = e^{2-x^2}$$

Find  $h'(x)$

**Solution:** Consider the composite of the functions

$$f(u) = e^u \quad \text{and} \quad g(x) = 2 - x^2$$

The derivatives of both  $f$  and  $g$  are

$$f'(u) = e^u \quad \text{and} \quad g'(x) = -2x$$

From the **chain rule**

$$h'(x) = e^{2-x^2}(-2x)$$

# Chain Rule for Special Functions

- **General Derivative of Exponential Function**

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$$

- **General Derivative of Logarithm Function**

$$\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$$

- **General Derivative of Sine**

$$\frac{d}{dx} \sin(f(x)) = f'(x) \cos(f(x))$$

- **General Derivative of Cosine**

$$\frac{d}{dx} \cos(f(x)) = -f'(x) \sin(f(x))$$

## Example 3: Derivative of Cosine Function

**Example 3:** Consider the function

$$f(x) = e^{-3x} \cos(x^2 + 4)$$

Find the derivative of  $f(x)$

Skip Example

**Solution:** This derivative uses the product and chain rule

$$f'(x) = e^{-3x}(-2x \sin(x^2 + 4)) + \cos(x^2 + 4)(e^{-3x}(-3))$$

$$f'(x) = -e^{-3x}(2x \sin(x^2 + 4) + 3 \cos(x^2 + 4))$$

## Example 4: More Examples of Differentiation

**Example 4:** Consider the function

$$f(x) = 3x^2 \sin(\ln(x + 2))$$

Find the derivative of  $f(x)$

Skip Example

**Solution:** This derivative uses the product and chain rule

$$f'(x) = (3x^2) \left( \frac{d}{dx} \sin(\ln(x + 2)) \right) + 6x \sin(\ln(x + 2))$$

$$f'(x) = \frac{3x^2 \cos(\ln(x + 2))}{x + 2} + 6x \sin(\ln(x + 2))$$

## Example 5: More Examples of Differentiation

**Example 5:** Consider the function

$$f(x) = 4e^{-\cos(2x+1)}$$

Find the derivative of  $f(x)$

Skip Example

**Solution:** This derivative uses the chain rule

$$f'(x) = 4e^{-\cos(2x+1)}(2 \sin(2x + 1))$$

# Rate of Change in Weight

**Rate of Change in Weight:** The example for the weight and height of a child given above found

- The weight  $W$  as a function of height  $h$  is

$$W(h) = 0.000720 h^{2.17}$$

- The height as a function of age is

$$h(a) = 6.45 a + 73.9$$

- The composite function weight as a function of age

$$W(a) = 0.000720(6.45 a + 73.9)^{2.17}$$

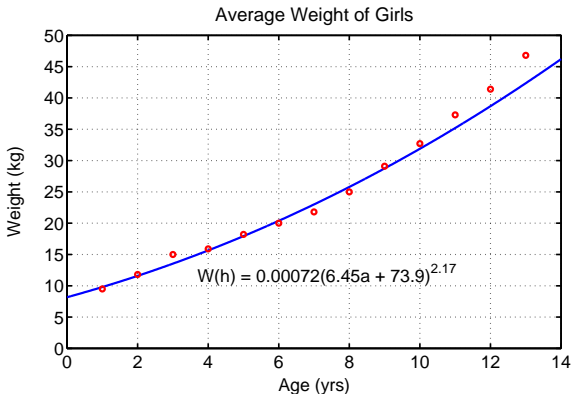


## Rate of Change in Weight

2

**Composite Function:** Weight as a function of age

$$W(a) = 0.000720(6.45a + 73.9)^{2.17}$$



## Rate of Change in Weight

**Chain Rule:** Weight as a function of age

$$W(a) = 0.000720(6.45 a + 73.9)^{2.17}$$

From the chain rule, the derivative of the weight function is

$$\frac{dW}{da} = \frac{dW}{dh} \cdot \frac{dh}{da}$$

$$\frac{dW}{dh} = 2.17(0.000720)h^{1.17} \quad \text{and} \quad \frac{dh}{da} = 6.45$$

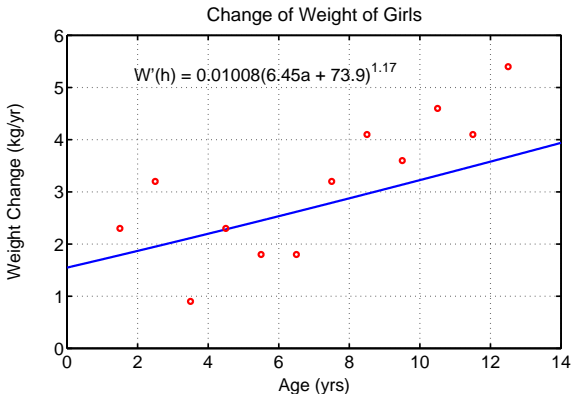
Combining these and substituting the expression for  $h$

$$W'(a) = 0.01008(6.45 a + 73.9)^{1.17}$$

## Rate of Change in Weight

## Change of Weight of Girls Graph of

$$W'(a) = 0.01008(6.45a + 73.9)^{1.17}$$



## Rate of Change in Weight

## Change of Weight of Girls Graph of

$$W'(a) = 0.01008(6.45a + 73.9)^{1.17}$$

- This graph is almost linear, since it is to the 1.17 power
- The actual average weight changes are given for the data above
- We see that the model under predicts the weight gain for older girls

# Normal Distribution

1

**Normal Distribution:** This is an important function in **statistics**

- Gives the classic **Bell curve**
- The **normal distribution function** is

$$N(x) = \frac{a}{\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

- $a$  is the normalizing factor
- $\sigma$  is the **standard deviation**
- $\mu$  is the **mean** of the distribution

# Normal Distribution

**Normal Distribution:** Consider

$$N(x) = \frac{a}{\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

- Find the maximum and points of inflection
- Plot this function for several values of  $\sigma$
- Discuss the importance of the results

## Normal Distribution

**Solution:** Consider

$$N(x) = \frac{a}{\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

The derivative is

$$\begin{aligned}\frac{dN}{dx} &= \frac{a}{\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \left( -\frac{2(x-\mu)}{2\sigma^2} \right) \\ \frac{dN}{dx} &= -\frac{a(x-\mu)}{\sigma^3} e^{-(x-\mu)^2/(2\sigma^2)}\end{aligned}$$

The derivative is zero when  $x = \mu$ , so there is a maximum at  $(\mu, \frac{a}{\sigma})$

## Normal Distribution

**Solution:** The derivative is

$$\frac{dN}{dx} = -\frac{a(x - \mu)}{\sigma^3} e^{-(x-\mu)^2/(2\sigma^2)}$$

The second derivative is

$$\frac{d^2N}{dx^2} = \frac{a}{\sigma^3} \left( (x - \mu) e^{-(x-\mu)^2/(2\sigma^2)} \left( -\frac{2(x - \mu)}{2\sigma^2} \right) + e^{-(x-\mu)^2/(2\sigma^2)} \cdot 1 \right)$$

$$\frac{d^2N}{dx^2} = \frac{a}{\sigma^3} \left( 1 - \frac{(x - \mu)^2}{\sigma^2} \right) e^{-(x-\mu)^2/(2\sigma^2)}$$

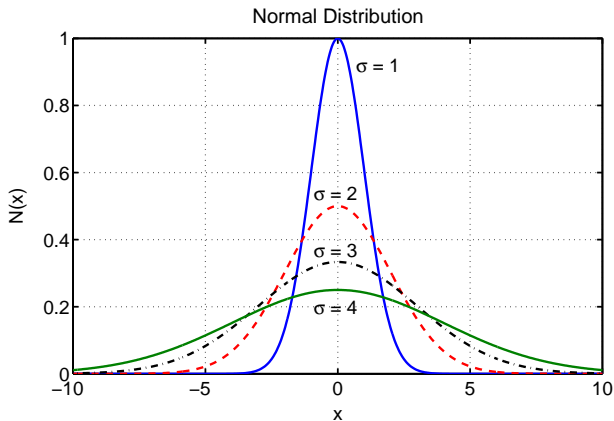
The points of inflection occur at  $x = \mu \pm \sigma$  with

$$N(\mu \pm \sigma) = \frac{a}{\sigma} e^{-\frac{1}{2}}$$



## Normal Distribution

**Solution:** Graph of the **Normal Distribution** with  $\mu = 0$  and  $\sigma = 1, 2, 3, 4$



# Normal Distribution

## Properties of Normal Distribution

$$N(x) = \frac{a}{\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

- As noted above, the **mean** of the normal distribution is  $\mu$
- The normal distribution is a **bell-shaped** curve centered about its mean
- The points of inflection occur one standard deviation,  $\sigma$ , from the mean,  $\mu$
- It can be shown that 68% of the area under the normal distribution occurs in the interval,  $[-\sigma, \sigma]$
- The **area** under a **distribution function** is important in measuring **probabilities** and **confidence intervals** for statistics

# Hassell's Model

**Hassell's Model** is often used in the study of insect populations

Consider **Hassell's Updating function**:

$$H(P) = \frac{81P}{(1 + 0.002P)^4}$$

- Find the intercepts and any asymptotes for  $P \geq 0$
- Find the derivative of  $H(P)$  and determine all extrema
- Graph  $H(P)$  for  $P \geq 0$

## Hassell's Model

**Solution:** Hassell's model satisfies:

$$H(P) = \frac{81P}{(1 + 0.002P)^4}$$

- For  $H(P)$ , the only intercept is  $(0, 0)$
- The power of  $P$  in the denominator (4) exceeds the power in the numerator (1), so there is a horizontal asymptote with  $H = 0$

## Hassell's Model

**Solution: (cont)** Next we find the derivative of Hassell's model:

$$H(P) = \frac{81P}{(1 + 0.002P)^4}$$

$$\frac{dH}{dP} = 81 \frac{(1 + 0.002P)^4 \cdot 1 - P \cdot 4(1 + 0.002P)^3 \cdot 0.002}{(1 + 0.002P)^8}$$

$$\frac{dH}{dP} = 81 \frac{(1 + 0.002P)^3(1 + 0.002P - 0.008P)}{(1 + 0.002P)^8}$$

$$\frac{dH}{dP} = 81 \frac{(1 - 0.006P)}{(1 + 0.002P)^5}$$

## Hassell's Model

**Solution (cont):** The derivative is

$$\frac{dH}{dP} = 81 \frac{(1 - 0.006 P)}{(1 + 0.002 P)^5}$$

- **Critical points** satisfy  $H'(P) = 0$ , so

$$1 - 0.006 P = 0 \quad \text{or} \quad P = \frac{500}{3} = 166.7$$

- With  $H(500/3) = 4271.5$ , the maximum occurs at

$$(166.7, 4271.5)$$

## Hassell's Model

**Solution (cont):** The graph of

$$H(P) = \frac{81P}{(1 + 0.002P)^4}$$

