1. (20 pts) mathbioLibrary/setABioc2Labs/Lab122_D3_fourier_fur.pg
Because of the accuracy of WebWork, you should use 5 or 6 significant figures on this problem.

In the Introduction to the lecture notes, we referred to the data from the Hudson Bay Company. This graph suggests that the population of snowshoe hares follows a periodic function, although it is not a true sine or cosine wave, due to their interaction with their primary predator the lynx. We noted in the beginning of the lecture section on differentiation of trigonometric functions that the work of Fourier showed that any continuous, periodic function can be modeled with the infinite summation of a series of sine and/or cosine waves.

In this question you will use the work of Fourier and Excel's solver, to model the quantity of the hare pelts (in thousands) turned into the Hudson Bay Trading Company for years after 1900. The data for hare pelts (in thousands) is shown below:

Year	Hare Pelts	Year	Hare Pelts	Year	Hare Pelts
1900	30	1907	21.4	1914	52.3
1901	47.2	1908	22	1915	19.5
1902	70.2	1909	25.4	1916	11.2
1903	77.4	1910	27.1	1917	7.6
1904	36.3	1911	40.3	1918	14.6
1905	20.6	1912	57	1919	16.2
1906	18.1	1913	76.6	1920	24.7

We would like to approximate these data with a model composed of a sum of sine functions of the form

$$P(t) = a_0 + \sum_{n=1}^{N} a_i \sin(n\omega(t-\phi_n)).$$

where $t = 0$ corresponds to 1900. We need to find the appropriate constants a_n, ω, and ϕ_n, which best fit the data. **Choose the unique parameters amplitude, $a_i > 0$, frequency, $\omega > 0$, and principle phase shift, $\phi_i \in [0, T)$, where T is the period of the appropriate trigonometric function.** Note that Excel's Solver may select $a_i < 0$ or ϕ_i outside the interval $[0, T)$. Use the periodic properties of the trigonometric functions to obtain the unique parameters described above.

a. Begin by trying to fit the data with a simple sine function and a constant, which satisfies the equation

$$P(t) = a_0 + a_1 \sin(\omega(t-\phi_1)),$$

where $t = 0$ corresponds to 1900. Use Excel's solver to find the constants a_0, a_1, ω, and ϕ_1, which give a least squares best fit to the data. For initial guesses, take a_0 to be the average of the data and a_1 be the difference between the maximum of the data and the average. Use the difference in years between the two maxima in the data to approximate the period and use that to estimate ω. Let $\phi_1 = 0$.

a_0 = _______
a_1 = _______
ω = _______
ϕ_1 = _______
Least Sum Square Errors = _______

What is the period of this approximation?

Period = _______

Find the percent error between the model and the data for $t = 4$ (1904), $t = 9$ (1909), $t = 13$ (1913), and $t = 18$ (1918).

$P(4)$ = _______ with Percent Error = _______
$P(9)$ = _______ with Percent Error = _______
$P(13)$ = _______ with Percent Error = _______
$P(18)$ = _______ with Percent Error = _______

Find the values of t, where the absolute minimum and absolute maximum occur, and values $P(t)$ of the absolute minimum and the absolute maximum from this approximation. (Give only the first occurrence as it is periodic.)

Minimum at t = _______ with $P(t_{min})$ = _______
Maximum at t = _______ with $P(t_{max})$ = _______

b. The next step in this problem is to see how much better the data are fit using another sine function in the Fourier series. The second sine function has double the frequency of the first (sharing ω). Thus, we want to fit the function

$$P(t) = a_0 + a_1 \sin(\omega(t-\phi_1)) + a_2 \sin(2\omega(t-\phi_2)).$$

Use Excel's solver to find the constants a_0, a_1, a_2, ω, ϕ_1, and ϕ_2, which give a least squares best fit to the data. For initial guesses, use your best fitting values of a_0, a_1, ω, and ϕ_1 from Part a. Take initial guesses of $a_2 = 1$ and $\phi_2 = 0$.

a_0 = _______
a_1 = _______
ω = _______
ϕ_1 = _______
a_2 = _______
ϕ_2 = _______
Least Sum Square Errors = _______

What is the period of this approximation?

Period = _______

Find the percent error between the model and the data for $t = 4$ (1904), $t = 9$ (1909), $t = 13$ (1913), and $t = 18$ (1918).

$P(4)$ = _______ with Percent Error = _______
$P(9)$ = _______ with Percent Error = _______
$P(13)$ = _______ with Percent Error = _______
$P(18)$ = _______ with Percent Error = _______

Find the values of t, where the absolute minimum and absolute maximum occur, and values $P(t)$ of the absolute minimum and the absolute maximum from this approximation. (Give only the first occurrence as it is periodic.)

Minimum at t = _______ with $P(t_{min})$ = _______
Maximum at t = _______ with $P(t_{max})$ = _______

c. Repeat the process in Part b. adding a third term,

$$P(t) = a_0 + a_1 \sin(\omega(t-\phi_1)) + a_2 \sin(2\omega(t-\phi_2)) + a_3 \sin(3\omega(t-\phi_3)).$$

Use Excel's solver to find the constants a_0, a_1, a_2, a_3, ω, ϕ_1, ϕ_2 and ϕ_3, which give a least squares best fit to the data. For initial guesses, use your best fitting values of a_0, a_1, a_2, ω, ϕ_1 and ϕ_2 from Part b. Take initial guesses of $a_3 = 1$ and $\phi_3 = 0$.

a_0 = _______
a_1 = _______
ω = _______
ϕ_1 = _______
a_2 = _______
ϕ_2 = _______
a_3 = _______
ϕ_3 = _______
Least Sum Square Errors = _______

What is the period of this approximation?

Period = _______

Find the percent error between the model and the data for $t = 4$ (1904), $t = 9$ (1909), $t = 13$ (1913), and $t = 18$ (1918).

$P(4)$ = _______ with Percent Error = _______
$P(9)$ = _______ with Percent Error = _______
$P(13)$ = _______ with Percent Error = _______
$P(18)$ = _______ with Percent Error = _______

Find the values of t, where the absolute minimum and absolute maximum occur, and values $P(t)$ of the absolute minimum and the absolute maximum from this approximation. (Give only the first occurrence as it is periodic.)

Minimum at t = _______ with $P(t_{min})$ = _______
Maximum at t = _______ with $P(t_{max})$ = _______

d. Repeat the process in Part c. adding a fourth term term, $a_4 \sin(4\omega(t-\phi_4))$.

Use Excel's solver to find the constants a_0, a_1, a_2, a_3, a_4 ω, ϕ_1, ϕ_2,ϕ_3 and ϕ_4, which give a least squares best fit to the data. For initial guesses, use your best fitting values of a_0, a_1, a_2, a_3, ω, ϕ_1, ϕ_2 and ϕ_3 from Part c. Take initial guesses of $a_4 = 1$ and $\phi_4 = 0$.

a_0 = _______
a_1 = _______
ω = _______
ϕ_1 = _______
a_2 = _______
ϕ_2 = _______
a_3 = _______
ϕ_3 = _______
a_4 = _______
ϕ_4 = _______
Least Sum Square Errors = _______

What is the period of this approximation?

Period = _______

Find the percent error between the model and the data for $t = 4$ (1904), $t = 9$ (1909), $t = 13$ (1913), and $t = 18$ (1918).

$P(4) =$ ________ with Percent Error = ________
$P(9) =$ ________ with Percent Error = ________
$P(13) =$ ________ with Percent Error = ________
$P(18) =$ ________ with Percent Error = ________

Find the values of t, where the absolute minimum and absolute maximum occur, and values $P(t)$ of the absolute minimum and the absolute maximum from this approximation. (Give only the first occurrence as it is periodic.)

Minimum at $t =$ ________ with $P(t_{min}) =$ ________
Maximum at $t =$ ________ with $P(t_{max}) =$ ________

e. In your Lab Report, on a single graph plot the data and all four models from $t \in [0,20]$, (1900 to 1920). Describe in some detail how each of the four models fits the data. Discuss the evolution from the first model to the fourth model on how the models compare to the data and what changes are observed in the parameters. Describe relative changes in the shared parameters for the sequence of models, such as the evolution of the amplitude, frequency, or phase shift. Do these models have the same period? What is the relative size (think amplitude) of the new terms in the subsequent models to the earlier models? That is discuss how much the different coefficients change as you add more terms to the series. Describe the shapes of the combined models and how well they fit the data. Write a brief discussion of What is the trend for the sum of least squares error and for the calculated percent error at the dates requested? Do you expect a much better fit by adding a fifth term to the series?

Write a short paragraph describing adjustments that you needed to obtain the correct parameters with Excel's Solver. Describe how you obtained the initial estimates, then explain how you made adjustments to obtain positive amplitudes and phase shifts in the appropriate range. In addition, describe the methods you used to find the absolute maxima and minima. Give details of the mathematical and numerical methods needed to find these critical points.

As described in the paragraph beginning this problem, these data come from a collection of pelts for snowshoe hares and lynx from the Hudson Bay trading company. These animals occupy a classical case of a tightly woven pair of prey animals and their primary predators. Write a paragraph on why these animal populations might exhibit an oscillatory relationship. Give some details on what you might expect on the relative temporal population changes of these populations. Specifically, take your data set for hare above and describe what you would expect for the population of the other animal, the lynx. Specifically, can you speculate the comparative relationships (in time) of the maxima and minima? Would the shape or period vary much between the population studies of these animals? Give a brief explanation.

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Eugene Odum, Fundamental of Ecology, Saunders, 1971.