# Stochastic Birth Process

This section examines a basic stochastic birth only process for a population. We assume that the probability of one birth is proportional to $\Delta t = \lambda\Delta t$, where $\Delta t$ is sufficiently small to allow only one birth event. With this assumption the probability of **not** giving birth is $1-\lambda\Delta t$. Let $P_N(t)$ be the probability of the population having $N$ individuals. Let $\nu_N$ be the probability that there are no births among $N$ individuals. Let $\sigma_{N-1}$ be the probability of one birth among $N-1$ individuals. Since there can be at most one birth in any time interval $\Delta t$ and this is a birth only process, then

$$P_N(t+\Delta t) = \sigma_{N-1}P_{N-1}(t) + \nu_N P_N(t).$$

We have

$$\nu_N = (1-\lambda\Delta t)^N \qquad \sigma_{N-1} \approx 1-(1-\lambda\Delta t)^{N-1}.$$

For small $\Delta t$ (keeping only linear terms in $\Delta t$ and ignoring higher order terms), we write:

$$\nu_N \approx 1-\lambda N\Delta t \qquad \sigma_{N-1} \approx \lambda(N-1)\Delta t.$$

If follows that

$$\begin{aligned} P_N(t+\Delta t) &\approx \lambda(N-1)\Delta t P_{N-1}(t) + (1-\lambda N\Delta t)P_N(t) \\ P_N(t+\Delta t) - P_N(t) &\approx \lambda(N-1)\Delta t P_{N-1}(t) - \lambda N\Delta t P_N(t) \\ \frac{P_N(t+\Delta t) - P_N(t)}{\Delta t} &\approx \lambda(N-1)P_{N-1}(t) - \lambda N P_N(t). \end{aligned}$$

Thus, we have the approximating differential equation for the birth only stochastic process given by the equation:

$$\frac{dP_N(t)}{dt} = \lambda(N-1)P_{N-1}(t) - \lambda N P_N(t),$$

which if we assume an initial population of $N_0$ individuals has the initial condition for the differential equation above is given by:

$$P_N(0) = \begin{cases} 0, & N \neq N_0 \\ 1, & N = N_0 \end{cases}$$

Let us consider the differential equation with $N = N_0$. Note that

$$P_{N_0-1}(t) = 0,$$

since this is a birth only process, and we are starting with a population of $N_0$ individuals. Thus, we want to satisfy the initial value problem given by:

$$\frac{dP_{N_0}(t)}{dt} = -\lambda N_0 P_{N_0}(t), \qquad P_{N_0}(0) = 1.$$

This has the solution

$$P_{N_0}(t) = e^{-\lambda N_0 t},$$

which is an exponentially decaying solution in time. As we would expect, as more time passes there is an increasing chance that a birth has occurred.

Now consider the differential equation with $N = N_0 + 1$. The new initial value problem is given by:

$$\frac{dP_{N_0+1}(t)}{dt} = \lambda N_0 P_{N_0}(t) - \lambda(N_0+1)P_{N_0+1}(t), \qquad P_{N_0+1}(0) = 0.$$

or

$$\frac{dP_{N_0+1}(t)}{dt} + \lambda(N_0+1)P_{N_0+1}(t) = \lambda N_0 e^{-\lambda N_0 t}, \qquad P_{N_0+1}(0) = 0.$$

This is a first order linear nonhomogeneous equation, which has the solution

$$P_{N_0+1}(t) = N_0 e^{-\lambda N_0 t}(1 - e^{-\lambda t}).$$

This solution begins with $P_{N_0+1}(0)$ being zero at $t = 0$ and asymptotically approaches zero as $t \to \infty$. Using techniques from Calculus, it is easily shown that there is a maximum at time

$$t_{max} = \frac{1}{\lambda}\ln\left(\frac{N_0+1}{N_0}\right),$$

which is the most likely time when the deterministic model reaches a population of $N_0+1$. The maximum value satisfies

$$P_{N_0+1}(t_{max}) = \left(\frac{N_0}{N_0+1}\right)^{N_0+1}.$$

This process continues to give the next probability distribution

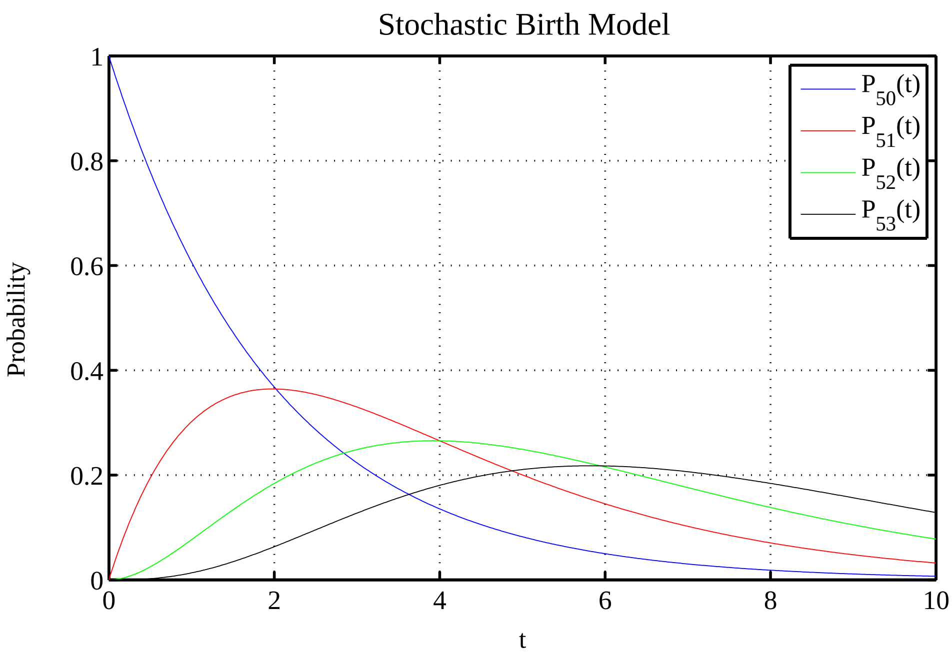
$$P_{N_0+2}(t) = \frac{N_0(N_0+1)}{2}e^{-\lambda N_0 t}(1 - e^{-\lambda t})^2.$$

Mathematical induction can be used to show that

$$P_{N_0+j}(t) = \frac{N_0(N_0+1)\cdot\ldots\cdot(N_0+j-1)}{j!}e^{-\lambda N_0 t}(1 - e^{-\lambda t})^j.$$

**Example**

We consider the birth only process with $\lambda = 0.01$ and $N_0 = 50$. The figure below shows the probability distributions for each of the first 4 populations, 50, 51, 52, and 53. The simulation begins with 50 individuals. As time passes, the chance of having exactly 50 individuals exponentially decreases. The probability of having exactly 51 individuals begins at zero, then rises to a peak probability at $t_{max} = 1.9803$ with probability $P_{51}(t_{max}) = 0.3642$. This probability distribution subsequently decays to zero as other population levels become more likely. Similarly, we can find the time where it is most likely that the population has 52 individuals, which occurs at $t_{max} = 3.92215$ with probability $P_{52}(t_{max}) = 0.2654$. These peaks are readily visible on the graph below. Each of the distributions, $P_{N_0+j}$ has its peak probability with larger times as $j$ increases. The distributions become broader and the peaks are lower as $j$ increases. At each time, $t$, the sum of all $P_{N_0+j}$ over $j$ is one.



Stochastic Birth Model
Probability
t
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10
P50(t)
P51(t)
P52(t)
P53(t)