I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

Give all answers to at least **4 significant figures**. This exam is due on **Friday, December 14** at the time of the final presentations.

1. The table below presents the population (in millions) for Mexico.

Year	Population
1950	28.49
1960	38.58
1970	52.78
1980	68.34
1990	84.91
2000	99.93

a. Find the growth rate for each decade with the data above. Associate each growth rate with the earlier of the two census dates. Determine the average (mean) growth rate, r, from the data above. Let t be the number of years after 1950, then associate t with the earlier of the dates in the growth ratio. Find the best straight line

$$k(t) = a + bt$$

through the growth data. Graph the constant function r, $k(t)$, and the data as a function of t over the period of the census data. Obtain at least 4 significant figures for a and b.

b. The Discrete Malthusian growth model is given by

$$P_{n+1} = (1 + r)P_n.$$

where r is computed in Part a, and P_0 is the population in 1950. Write the general solution to this model, where n is in decades. Use the model to predict the population in 2020 and 2050.

c. The revised growth model is given by

$$P_{n+1} = (1 + k(t_n))P_n.$$

where $k(t_n)$ is computed in Part a, $t_n = 10n$, and P_0 is again the population in 1950. Simulate this nonautonomous discrete dynamical model from 1950 to 2050. Use the model to predict the population in 2020 and 2050. This model predicts that the population will reach its maximum and start declining. Use the growth rate $k(t)$ to find when this model predicts a maximum population, then estimate what that maximum population will be.

d. Use the data above to find the best discrete logistic growth model fit for the population of Mexico. (Again use P_0 as the population in 1950 and only vary the growth rate and carrying capacity.) Add the graph of this model to your previous graph of the Malthusian growth models for the time period 1950 to 2050. What does this model predict for the population of Mexico in 2050? From the sum of square errors, which model matches the data best? Find all equilibria of this model and discuss the stability of these equilibria (include the values of the derivatives at the equilibria). What does this model predict will happen over a long period of time for Mexico's population?

e. Briefly discuss the strengths and weaknesses of each of these models. Which model does the best job of predicting the population into the future and for how long?

2. A. C. Crombie studied *Rhizopertha dominica*, the American wheat weevil, with an almost constant nutrient supply (maintained 10 g of cracked wheat weekly). These conditions match the assumptions of the discrete logistic model. The data below show the adult population of *Rhizopertha* from Crombie's study (with some minor modifications to fill in uncollected data and an initial shift of one week).

Week	Adults	Week	Adults
0	2	18	307
2	3	20	340
4	8	22	318
6	62	24	335
8	119	26	360
10	140	28	342
12	185	30	353
14	205	32	339
16	267	34	345

a. The discrete logistic growth model for the adult population P_n can be written

$$P_{n+1} = f(P_n) = rP_n - mP_n^2,$$

where the constants r and m must be determined from the data. An alternative model commonly used is the Beverton-Holt model given by

$$P_{n+1} = H(P_n) = \frac{aP_n}{1 + bP_n},$$

where the constants a and b must be determined from the data. Use the data above to find the best fitting updating functions for both models, giving the best constants r, m, a, and b along with the sum of square errors of the updating function to the data. Show a graph $f(P)$, $H(P)$, and the data along with the identity map, $P_{n+1} = P_n$.

b. Find the equilibria for these models. Write the derivatives of the updating functions. Discuss the behavior of each of the models near their equilibria.

c. Use the models found above to simulate the data. Find the best fitting model to the data by using the models found in Part a (the updating functions) and adjusting the initial condition, P_0, to give the least squares best fit to the data. (Adjust the initial conditions separately for each model.) Write both the sum of square errors and P_0 for each model. Graph these simulations and the data (adult population vs. time). What do you predict will happen to the adult American wheat weevil population for large times (assuming experimental conditions continue) according to each of these models? Discuss the similarities and differences that you observe between these models and how well they work for this experimental situation.

3. G. F. Gause in his book *The Struggle for Existence* studied a number of competition and predator-prey systems in the lab. One predator-prey system that he studied was the interaction between the predator *Didinium nasutum* and its prey *Paramecium caudatum.* Below is a table for one of his experiments.

Day t	*P. caud.* P	*D. nasu.* N	Day t	*P. caud.* P	*D. nasu.* N
0	1	1	9	3	13
1	5	1	10	1	6
2	8	2	11	7	2
3	17	2	12	16	1
4	24	5	13	27	4
5	56	8	14	18	22
6	26	31	15	3	36
7	9	26	16	2	15
8	6	15	17	1	8

a. Use the material from the lynx and hare study in class to find the best fitting parameters to the predator-prey model given by:

$$\begin{aligned}\dot{P} &= a_1P - a_2PN, \\ \dot{N} &= -b1N + b_2PN,\end{aligned}$$

where P is the population of *Paramecium caudatum* and N is the population of *Didinium nasutum.* Give your best fitting parameters $P(0)$, $N(0)$, a_1, a_2, b_1, and b_2 along with the sum of square errors to the data. (Hint: This problem converges poorly for the wrong set of parameters, so you may want to either adjust a few parameters by hand until you are closer to matching the data or fix a few parameters and adjust only 2-4 others to get closer to the correct answer.)

b. Find all equilibria for this model, then discuss the stability of these equilibria, giving the eigenvalues. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral). Is this model structurally stable? Produce a graph with the time evolution of both populations and another graph showing the phase portrait of the two populations (including arrows to show the direction of the solution). Be sure to include the data on your graph.

c. We also examined an alternative predator-prey model, where the model includes an intraspecies competition term for the prey species. This model is given by the system of differential equations

$$\begin{aligned}\dot{P} &= a_1P - a_2PN - a_3P^2, \\ \dot{N} &= -b1N + b_2PN.\end{aligned}$$

Repeat the process that you did in Parts a and b to find the new best set of parameters and its sum of square errors to the data. Find all equilibria with their eigenvalues, and characterize these equilibria. Is this model structurally stable? Once again, produce a graph with the time evolution of both populations and another graph showing the phase portrait of the two populations (including arrows to show the direction of the solution). Be sure to include the data on your graph. Briefly discuss which model appears to be better and why.

4. A common model used by fisheries is Ricker's model given by

$$\frac{dN}{dt} = N(re^{-\beta N} - 1),$$

where r and β are positive parameters determined by the ecosystem.

a. Find the equilibria for this model and determine the stability of these equilibria.

b. Let $r = 4$ and $\beta = 0.005$. Consider the two Ricker's models with fishing given by:

$$\begin{aligned}\frac{dN}{dt} &= N(re^{-\beta N} - 1) - h_1,\\ \frac{dN}{dt} &= N(re^{-\beta N} - 1) - h_2 N,\end{aligned}$$

where $h_1 \geq 0$ is a constant level of fishing and $h_2 \geq 0$ is a level of fishing proportional to the population of fish.

c. Find the maximum level of harvesting (value of h_1 or h_2) using each of these methods of fishing that allows the population of fish to survive. Also, determine the maximum number of fish that can be harvested from each of these fishing techniques. Also, create a bifurcation diagram for each of these fishing models using either h_1 or h_2 as your bifurcation parameter. Be sure to state what type of bifurcation occurs in each case.

5. An age-structured population of birds was surveyed over 4 years. The researchers determined the number of birds in each age class for each of the 4 years and found out how many nestlings fledged from each of the different age classes each year. The researchers divided the population of birds into the birds 0-1 years old, 1-2 years old, and those that are older. This age-structured population forms a Leslie model of the following form:

$$\begin{pmatrix} P_1(n+1) \\ P_2(n+1) \\ P_3(n+1) \end{pmatrix} = \begin{pmatrix} 0 & b_2 & b_3 \\ s_{12} & 0 & 0 \\ 0 & s_{23} & s_{33} \end{pmatrix} \begin{pmatrix} P_1(n) \\ P_2(n) \\ P_3(n) \end{pmatrix}.$$

a. The table below shows how many birds in each age class survived to the next year (and gives the total number of birds that fledged). The researchers determined that the survival of the 1-2 year old birds is roughly equal to the survival of the older birds. Thus, we can assume that $s_{23} = s_{33}$. Use the data below to compute the average values for each of the survival parameters s_{12} and $s_{23} = s_{33}$.

Bird Age	Year 1	Year 2	Year 3	Year 4
0-1	175	237	258	311
1-2	42	59	89	92
older	97	104	128	145

They also collected data on the success rate of nesting of each of the different age classes of birds. The table below shows the number of fledglings raised by each of the age classes over the 4 year period. (Note that these columns total to the number of 0-1 year old birds the next year.) Use the data below to compute the average birth rates for each of the age classes b_2 and b_3. (One year old birds of this species don't nest.)

Bird Age	Year 1	Year 2	Year 3	Year 4
1-2	38	47	66	74
older	199	211	245	293

b. Write the Leslie matrix for this species of bird using the average values computed above (to **4 significant figures**). Use your Leslie matrix to estimate the population of each of the age classes for the next 3 years. (Use the last surveyed data as your starting point for this simulation.)

c. Find the eigenvalues and eigenvectors for this model, then give the limiting percent population in each of the age classes. What is the approximate annual rate of growth for this species of bird and how long would it take for the total population to double?

6. An enclosed area is divided into four regions with varying habitats. One hundred tagged frogs are released into the first region. Earlier experiments found that on average the movement of frogs each day about the four regions satisfied the transition model given by

$$\begin{pmatrix} f_1(n+1) \\ f_2(n+1) \\ f_3(n+1) \\ f_4(n+1) \end{pmatrix} = \begin{pmatrix} 0.42 & 0.16 & 0.19 & 0.16 \\ 0.07 & 0.38 & 0.24 & 0.13 \\ 0.34 & 0.19 & 0.51 & 0.27 \\ 0.17 & 0.27 & 0.06 & 0.44 \end{pmatrix} \begin{pmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \\ f_4(n) \end{pmatrix}.$$

a. Give the expected distribution of the tagged frogs after 1, 2, 5, and 10 days.

b. What is the expected distribution of the frogs after a long period of time? Which of the four regions is the most suitable habitat and which is the least suitable for these frogs?

c. These transitions are random events. Write a MatLab code for a Monte Carlo simulation of this experiment. (Show your code with comments to explain what the code is doing!) Run the experiment 1000 times, giving the mean and standard deviation of the distribution of frogs after 1, 2, 5, and 10 days. Compare these results to Part a.