Due December 1, 2004 (11 AM) Math 342A Name ____________________

1. Solve the following initial value problems.

a. $(t^2 - \sin(y))\dfrac{dy}{dt} = e^t - 2ty, \quad y(2) = 0$

b . $\dfrac{dy}{dt} = e^{-t}\left(1 + y^2\right), \quad y(0) = 0$

c. $\dfrac{dy}{dt} = t^3 - 2ty, \quad y(0) = 10$

d. $\dfrac{dy}{dt} = \dfrac{y^2 - 2ty}{t^2}, \quad y(1) = 2$

2. Consider the following initial value problem:

$$\frac{dy}{dt} = y - t^2, \qquad y(0) = 1.$$

a. Solve this differential equation.

b. Show the slope field for $t \in [-5, 5]$ and $y \in [-5, 5]$, including the solution of the initial value problem.

c. Use Eulers method and the 4^{th} order Runge-Kutta to simulate the solution for $t \in [0, 5]$ with a stepsize of $h = 0.2$. Create a table comparing the actual solution to each of these numerical simulations at times $t = 1, 2, 3, 4,$ and 5. Include the percent error between the numerical and actual solution. (Do **NOT** write all your calculations needed for the numerical solutions!) Graph the actual solution and each of the numerical simulations.

3. a. The normal body temperature of a cat is 39°C. A cat is hit by a car at some time during the night. When it is discovered at 6 AM, the young scientist discovers that the body temperature of the cat is 31°C. (Let this be your initial condition, $T(0) = 31$.) Assume that the body of the cat satisfies Newton's Law of Cooling, which is given by the differential equation:

$$\frac{dT}{dt} = -k\left(T - T_e\right),$$

where T_e is the environmental temperature. If the early morning temperature satisfies $T_e = 15$°C, then solve this initial value problem.

b. Two hours later the temperature of the body is found to be 29.2°C. Find the constant of cooling k, and determine when the death occurred.

c. Since it is early morning, the temperature has been decreasing for some length of time rather than remaining constant. Suppose that a more accurate cooling law uses the above differential equation with $T_e = 15 - t/2$ instead of just a constant. Solve this new differential equation. Find the new constant k and approximate the time of death using the information above with this new differential equation.

4. a. The U. S. population was 76.0 million in 1900 and 105.7 million in 1920. Use the Malthusian growth model $P' = rP$ to represent the population of the U. S. Solve this differential equation, assuming that $t = 0$ is 1900 and with the data above find the value of r to 4 significant figures. How long does it take for the population to double with this model?

b. The population of U. S. was 92.0 million in 1910 and 151.3 million in 1950. Find the percent error between the Malthusian growth model predictions and these data.

c. A logistic growth model for the U. S. population that reasonably fits the census data for the 20^{th} century is given by

$$\frac{dP}{dt} = 0.02P\left(1 - \frac{P}{420}\right), \qquad P(0) = 76.0.$$

where t is in years after 1900. Solve this differential equation, then find the percent error between the actual population and this model for the years 1910, 1920, and 1950. How long does it take for the 1900 population to double with this model?

d. Determine the equilibria for this logistic growth model. At what population does this model predict the U. S. will level off?

5. a. Let x_1 be the amount of a radioactive substance with a half-life of 4 years and assume that you begin with 10 g of the radioactive material. Write a differential equation for x_1 (including the initial condition) and solve it. Give your decay constant to at least 4 significant figures. Determine how long until 8 g remains.

b. Suppose x_1 decays into another radioactive substance x_2 with a half-life of 64 years. The amount of radioactive material x_2 satisfies the differential equation:

$$\dot{x}_2 = -k_2 x_2 + k_1 x_1,$$

where x_1 is the solution and k_1 is the decay constant from part a. and k_2 is found from the half-life of this second radioactive material. Solve this equation if initially there is no x_2. Determine the time when the most amount of x_2 is available and how much of x_2 is present at that time.