1. a. The first integral is written as a power, while the second integral uses the substitution $u = 3x - 2$, so $du = 3\, dx$.

$$\int (3x^{-2} + 3\cos(3x - 2))\, dx = 3 \int x^{-2}\, dx + \int \cos(u)\, du$$

$$= 3\frac{x^{-1}}{-1} + \sin(u) + C = -\frac{3}{x} + \sin(3x - 2) + C$$

b. Let $u = -x^2$, so $du = -2\, dx$.

$$\int (2x e^{-x^2} - 4x)\, dx = -\int e^u\, du - \frac{4x^2}{2}$$

$$= -e^u - 2x^2 + C = -e^{-x^2} - 2x^2 + C$$

c. Let $u = 4 + e^{-2x}$, so $du = -2e^{-2x}\, dx$.

$$\int \frac{4e^{-2x}}{\sqrt{4 + e^{-2x}}}\, dx = -2\int u^{-\frac{1}{2}}\, du$$

$$= -4u^{\frac{1}{2}} + C = -4\sqrt{4 + e^{-2x}} + C$$

d. Let $u = 3 + \cos(x^2 + 1)$, so $du = -2x\sin(x^2 + 1)\, dx$.

$$\int \frac{x \sin(x^2 + 1)}{3 + \cos(x^2 + 1)}\, dx = -\frac{1}{2} \int \frac{du}{u}$$

$$= -\frac{1}{2} \ln|u| + C = -\frac{1}{2} \ln(3 + \cos(x^2 + 1)) + C$$

e. Both antiderivatives are standard for this problem

$$\int_0^{2\pi} (\cos(t/4) + t)\, dt = \left(4\sin(t/4) + \frac{t^2}{2}\right)|_t=0$$

$$= 4\sin(\pi/2) + \frac{4\pi^2}{2} - 4\sin(0) - 0 = 4 + 2\pi^2$$

f. The first integral uses the substitution $u = x^2 + 1$, so $du = 2\, dx$ with endpoints $u = 2$ when $x = 1$ and $u = 5$ when $x = 2$, while the second antiderivative is the natural logarithm

$$\int_1^2 \left(\frac{6x}{x^2 + 1} + \frac{2}{x}\right)\, dx = 3 \int_2^5 \frac{du}{u} + 2\ln|x|^2$$

$$= 3\ln|u|^5|_{u=2} + 2 \ln(2) - 2 \ln(1)$$

$$= 3\ln(5) - 3\ln(2) + 2 \ln(2) = 3\ln(5) - \ln(2) = \ln(62.5) \simeq 4.135$$
g. Let \( u = 25 - x^2 \), so \( du = -2x dx \) with endpoints \( u = 25 \) when \( x = 0 \) and \( u = 0 \) when \( x = 5 \).

\[
\int_{0}^{5} x \sqrt{25 - x^2} \, dx = -\frac{1}{2} \int_{25}^{0} u^{\frac{1}{2}} \, du \\
= -\frac{1}{2} \left( \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) \bigg|_{u=25}^{0} \\
= -\frac{1}{3} \left( 0^{\frac{3}{2}} - 25^{\frac{3}{2}} \right) = \frac{125}{3}
\]

h. For the first integral, let \( u = \ln(x) \), so \( du = dx/x \) with endpoints \( u = 0 \) when \( x = 1 \) and \( u = \ln(4) \) when \( x = 4 \). The second integral is a power rule.

\[
\int_{1}^{4} \left( \frac{\ln(x)}{x} + \frac{3}{\sqrt{x}} \right) \, dx = \int_{0}^{\ln(4)} u \, du + 3 \int_{1}^{4} x^{-\frac{1}{2}} \, dx \\
= \frac{u^{2}}{2} \bigg|_{u=0}^{\ln(4)} + 6 \sqrt{x} \bigg|_{1}^{4} \\
= \frac{1}{2} (\ln(4))^2 + 6(2 - 1) = 6 + 2 (\ln(2))^2
\]

2. a. This is a separable differential equation. It can be written

\[
\int 2y \, dy = \int 3t^2 \, dt \quad \text{or} \quad y^2 = t^3 + C.
\]
It follows that \( y(t) = \pm \sqrt{t^3 + C} \). The initial condition \( y(0) = 4 = \sqrt{C} \), which implies \( C = 16 \). Hence, the solution is

\[
y(t) = \sqrt{t^3 + 16}.
\]

b. This is a linear differential equation, which can be written

\[
\frac{dy}{dt} = -0.02(y - 100).
\]
With the substitution \( z(t) = y(t) - 100 \), we have

\[
\frac{dz}{dt} = -0.02z, \quad z(0) = y(0) - 100 = -95.
\]
Thus, \( z(t) = -95 e^{-0.02t} \). It follows that

\[
y(t) = 100 - 95 e^{-0.02t}.
\]
c. This is a separable differential equation. It can be written
\[ \int \frac{dy}{y} = \int \frac{2t \, dt}{t^2 + 1}. \]
The right integral uses the substitution \( u = t^2 + 1 \), so \( du = 2t \, dt \). Hence,
\begin{align*}
\ln |y(t)| & = \int \frac{du}{u} = \ln |u| + C = \ln(t^2 + 1) + C \\
y(t) & = e^{\ln(t^2+1)+C} = A(t^2 + 1),
\end{align*}
where \( A = e^C \). The initial condition \( y(0) = 3 = A \), which implies \( A = 3 \). Hence, the solution is
\[ y(t) = 3(t^2 + 1). \]

d. This is a separable differential equation. It can be written
\[ \int \frac{dy}{y} = \int (2 - 0.2t) \, dt \quad \text{or} \quad \ln |y| = 2t - 0.1 t^2 + C. \]
It follows that \( y(t) = e^{2t-0.1t^2+C} = A e^{2t-0.1t^2} \) with \( A = e^C \). The initial condition \( y(0) = 10 = A \), which implies \( A = 10 \). Hence, the solution is
\[ y(t) = 10 e^{2t-0.1t^2}. \]

e. This is a time-varying differential equation, so we integrate giving
\[ y(t) = \int (4 - 2 \sin(2(t - 3))) \, dt = 4t - 2 \int \sin(2(t - 3)) \, dt. \]
With the substitution \( u = 2(t - 3) \) and \( du = 2 \, dt \), we have
\[ y(t) = 4t - \int \sin(u) \, du = 4t + \cos(u) + C = 4t + \cos(2(t - 3)) + C. \]
With the initial condition \( y(3) = 5, 12 + \cos(0) + C = 5 \) or \( C = -8 \). It follows that
\[ y(t) = 4t + \cos(2(t - 3)) - 8. \]

f. This is a separable differential equation. It can be written
\[ \int e^y \, dy = \int e^t \, dt \quad \text{or} \quad e^y = e^t + C. \]
It follows that \( y(t) = \ln(e^t + C) \). The initial condition \( y(0) = 6 = \ln(1 + C) \), which implies \( C = e^6 - 1 \). Hence, the solution is
\[ y(t) = \ln(e^t + e^6 - 1). \]
3. a. From the equation of the line, \( y = 3 - x \), the \( x \) and \( y \)-intercepts are easily seen to be \((3,0)\) and \((0,3)\), respectively. Also, the slope of the line is \( m = -1 \). The equation for the parabola is \( y = 6 + x - x^2 = -(x + 2)(x - 3) \). From this it is easy to see that the \( x \)-intercepts are \((-2,0)\) and \((3,0)\), while the \( y \)-intercept is \((0,6)\). The vertex of the parabola has its \( x \)-coordinate halfway between the \( x \)-intercepts, thus the vertex is \((1/2,6 1/4)\).

Setting the two equations together, \( 3 - x = 6 + x - x^2 \) or \( x^2 - 2x - 3 = (x + 1)(x - 3) = 0 \). Thus, the points of intersection are \((-1,4)\) and \((3,0)\). The graph can be seen below.

b. The area between the two curves is given by

\[
\int_{-1}^{3} (6 + x - x^2 - (3 - x)) \, dx = \int_{-1}^{3} 3 + 2x - x^2 \, dx
\]

\[
= \left[ 3x + \frac{x^3}{3} \right]_{x=-1}^{3}
\]

\[
= 9 + 9 - 9 + 3 - 1 - \frac{1}{3} = \frac{32}{3}
\]
4. a. The period of this function is $2\pi$. This is a typical cosine function with an amplitude of 2 and with the graph shifted up by 2 units. The graph can be seen below.

b. The area under the curve is given by

\[ \int_0^{2\pi} 2 \cos(t) + 2 \, dt = (2 \sin(t) + 2 \, t)|_{\pi}^{2\pi} = 4\pi \]

5. The differential equation is separable, so write

\[ \int T^{-\frac{1}{2}} \, dT = k \int dt \quad \text{or} \quad 2T^{\frac{1}{2}}(t) = kt + C. \]

It follows that

\[ T(t) = \left( \frac{kt + C}{2} \right)^2. \]

The initial condition $T(0) = 1$ implies $C = 2$, so $T(t) = \left( \frac{kt + 1}{2} \right)^2$. Since $T(4) = \left( \frac{4k}{2} + 1 \right)^2 = 25$, $2k + 1 = 5$ or $k = 2$. Thus, the solution for the spread of the disease in this orchard is

\[ T(t) = (t + 1)^2. \]

When $t = 10$, $T(10) = 121$. 

\[ \int T^{-\frac{1}{2}} \, dT = k \int dt \quad \text{or} \quad 2T^{\frac{1}{2}}(t) = kt + C. \]
6. a. The differential equation for Gompertz law of growth is separable. The solution can be found as follows, where we solve the complicated integral with the substitution of \( u = \ln \left( \frac{N}{2000} \right) \), so \( du = \frac{dN}{N} \).

\[
\frac{dN}{dt} = -0.1N \ln \left( \frac{N}{2000} \right)
\]

\[
\int \left( \ln \left( \frac{N}{2000} \right) \right)^{-1} \frac{dN}{N} = -0.1 \int dt = -0.1 \ t + C
\]

\[
\int u^{-1} du = \ln |u| = \ln \left( \ln \left( \frac{N(t)}{2000} \right) \right) = -0.1 \ t + C
\]

\[
\ln \left( \frac{N(t)}{2000} \right) = e^{-0.1t+C} = Ae^{-0.1t}
\]

\[
\frac{N(t)}{2000} = e^{Ae^{-0.1t}}
\]

\[
N(t) = 2000 e^{Ae^{-0.1t}}
\]

The initial condition is \( N(0) = 10 \), so \( 10 = 2000e^A \) or \( A = -\ln(200) \). Thus, the solution is given by

\[
N(t) = 2000 e^{-\ln(200)e^{-0.1t}}.
\]

b. For large time, \( e^{-0.1t} \to 0 \), so \( N(t) \to 2000e^0 = 2000 \) (in thousands). Hence, \( N(t) \to 2000 \) is a carrying capacity, which means that the tumor levels off with a population of 2 million cells.

7. a. The solution of the Mathusian growth model is \( B(t) = 1000 e^{0.01t} \). The population doubles when the bacteria reaches 2000, so \( 1000 e^{0.01t} = 2000 \) or \( e^{0.01t} = 2 \). Thus, \( 0.01t = \ln(2) \) or \( t = 100 \ln(2) \approx 69.3 \) min for the population to double.

b. The model with time-varying growth is a separable differential equation, so

\[
\frac{dB}{dt} = 0.01(1 - e^{-t}) B
\]

\[
\int \frac{dB}{B} = 0.01 \int (1 - e^{-t}) dt
\]

\[
\ln |B(t)| = 0.01(t + e^{-t}) + C
\]

\[
B(t) = Ae^{0.01(t+e^{-t})},
\]

where \( A = e^C \). With the initial condition, \( B(0) = 1000 = Ae^{0.01} \) or \( A = 1000 e^{-0.01} \). Thus, the solution to this time-varying growth model is

\[
B(t) = 1000 e^{0.01(t+e^{-t}-1)}.
\]

c. The Malthusian growth model gives \( B(5) = 1051 \) and \( B(60) = 1822 \), while the modified growth model gives \( B(5) = 1041 \) and \( B(60) = 1804 \).
8. a. The solution to the Malthusian growth model is given by \( P(t) = 100 e^{0.2t} \). This population doubles when \( 100 e^{0.2t} = 200 \) or \( e^{0.2t} = 2 \), so \( t = 5 \ln(2) \approx 3.466 \) yrs.

b. This model, including the modification for habitat encroachment, is a separable differential equation. It can be written

\[
\int \frac{dP}{P} = \int (0.2 - 0.02t) \, dt \quad \text{or} \quad \ln|P| = 0.2t - 0.01t^2 + C.
\]

It follows that \( P(t) = e^{0.2t - 0.01t^2} + C = Ae^{0.2t - 0.01t^2} \). The initial condition \( P(0) = 100 = A \), which implies \( A = 100 \). Hence, the solution satisfies

\[
P(t) = 100 e^{0.2t - 0.01t^2}.
\]

c. We find the maximum by differentiating and setting it equal to zero,

\[
P'(t) = 100 e^{0.2t - 0.01t^2} (0.2 - 0.02t) = 0.
\]

So \( 0.2 - 0.02t = 0 \), which implies that \( t = 10 \). Thus, the maximum of population is \( P(10) = 100 e \approx 271.8 \). If we solve \( P(t) = 100 e^{0.2t - 0.01t^2} = 100 \), then this is equivalent to \( e^{0.2t - 0.01t^2} = 1 \) or \( 0.2t - 0.01t^2 = \ln(100) = \ln(100) \quad t(t - 20) = 0 \). Thus, either \( t = 20 \) (or 0), so the population returns to 100 after 20 years. The graph of the population can be seen below.
9. a. This population of cells in a declining medium satisfies a separable differential equation, which can be written

\[ \int P^{-2/3} dP = \int 0.3 e^{-0.01t} dt \quad \text{or} \quad 3 P^{1/3}(t) = -30 e^{-0.01t} + 3C. \]

It follows that \( P^{1/3}(t) = -10 e^{-0.01t} + C \), so \( P(t) = (C - 10 e^{-0.01t})^3 \). The initial condition \( P(0) = 1000 = (C - 10)^3 \), which implies \( C = 20 \). The solution is given by

\[ P(t) = (20 - 10 e^{-0.01t})^3. \]

b. This population doubles when \( P(t) = (20 - 10 e^{-0.01t})^3 = 2000 \), so \( 20 - 10 e^{-0.01t} = 10 \sqrt[3]{2} \) or \( e^{-0.01t} = 2 - \sqrt[3]{2} \). It follows that \( t = 100 \ln \left( \frac{1}{2 - \sqrt[3]{2}} \right) \approx 30.1 \) hr. For large \( t \), \( \lim_{t \to \infty} e^{-0.01t} = 0 \), so \( \lim_{t \to \infty} P(t) = 20^3 = 8000 \). Thus, there is a horizontal asymptote at \( P = 8000 \), so the population tends towards this value. The graph of the population can be seen below.
10. a. The change in amount of phosphate, \( P(t) \), is found by adding the amount entering and subtracting the amount leaving.

\[
\frac{dP}{dt} = 200 \cdot 10 - 200 \cdot c(t),
\]

where \( c(t) \) is the concentration in the lake with \( c(t) = P(t)/10,000 \). By dividing the equation by the volume, the concentration equation is given by

\[
\frac{dc}{dt} = 0.2 - 0.02c = -0.02(c - 10), \quad c(0) = 0.
\]

With the substitution \( z(t) = c(t) - 10 \), the equation above reduces to the problem

\[
\frac{dz}{dt} = -0.02z, \quad z(0) = -10,
\]

which has the solution \( z(t) = -10 e^{-0.02t} \). Thus, the concentration is given by

\[
c(t) = 10 - 10 e^{-0.02t}.
\]

b. The differential equation describing the growth of the algae is given by

\[
\frac{dA}{dt} = 0.5(1 - e^{-0.02t})A^{2/3}.
\]

By separating variables, we see

\[
\int A^{-2/3}dA = 0.5 \int (1 - e^{-0.02t})dt
\]

\[
3 A^{1/3}(t) = 0.5(t + 50 e^{-0.02t}) + C
\]

\[
A(t) = \left( \frac{0.5(t + 50 e^{-0.02t}) + C}{3} \right)^3
\]

From the initial condition \( A(0) = 1000 \), we have \( 1000 = \left( \frac{25+C}{3} \right)^3 \). It follows that \( C = 5 \), so

\[
A(t) = \left( \frac{t + 50 e^{-0.02t} + 10}{6} \right)^3.
\]
11. a. The maximum population for $P(t) = 54 + 24t - 4t^2$ is found by differentiating with $P'(t) = 24 - 8t$, which is zero at $t = 3$. This gives a maximum population of $P(3) = 90$.

b. The maximum population for $Q(t) = 54 + 34 \sin \left( \frac{\pi}{6} t \right)$ occurs when $t = 3$, which is when the argument of the sine function is at $\pi/2$. This gives a maximum of $Q(3) = 88$. A graph of the function is below.

c. The average of the data is $\frac{54 + 73 + 85 + 89 + 86 + 75 + 53}{7} = 73.57$. The averages from the integrals are

$$P_{\text{ave}} = \frac{1}{6} \int_0^6 P(t) dt = \frac{1}{6} \int_0^6 54 + 24t - 4t^2 dt = \frac{1}{6} \left( 54t + 12t^2 - \frac{4}{3} t^3 \right) \bigg|_{x=0}^{x=6} = \frac{1}{6} (54(6) + 72(6) - 48(6)) = 78$$

and

$$Q_{\text{ave}} = \frac{1}{6} \int_0^6 Q(t) dt = \frac{1}{6} \int_0^6 54 + 34 \sin(1/6 \pi t) dt = \frac{1}{6} \left( 54t - \frac{204}{\pi} \cos(1/6 \pi t) \right) \bigg|_{x=0}^{x=6} = \frac{1}{6} \left( 54(6) - \frac{204}{\pi} \left( \cos(\pi) - \cos(0) \right) \right) = 54 + \frac{68}{\pi} = 75.65.$$
12. a. The Riemann sum using the data is given by

\[ \sum_{i=0}^{11} c(t_i) \Delta t = 0 \cdot 2 + 18 \cdot 2 + \ldots + 0 \cdot 2 = 246. \]

b. Computing the integral for the polynomial fit, we find

\[ \int_{0}^{20} (0.038 t^3 - 1.3 t^2 + 11 t) dt = \left( \frac{0.038 t^4}{4} - \frac{1.3 t^3}{3} + \frac{11 t^2}{2} \right)_{t=0}^{20} = 253.3. \]

c. Computing the integral for the exponential fit by letting \( u = -0.02 t^2 \) with \( du = -0.04 t \, dt \) and taking the endpoints \( u = 0 \) when \( t = 0 \) and \( u = -8 \) when \( t = 20 \), we find

\[ \int_{0}^{20} 10 t e^{-0.02 t^2} dt = -250 \int_{0}^{-8} e^u du = -250 e^{-8} |_{u=0} = 250(1 - e^{-8}) = 249.9. \]

d. The second model is better because it continues close to zero, while the polynomial tends to infinity. If a particular model is shown to be valid, then fewer blood samples would need to be drawn to get a reasonable fit to the data and approximate the drug dosage.

13. a. We apply the quotient rule and obtain

\[ f'(t) = \frac{(e^{10t} + e^{-10t})(10e^{10t} + 10e^{-10t}) - (e^{10t} - e^{-10t})(10e^{10t} - 10e^{-10t})}{(e^{10t} + e^{-10t})^2} = \frac{40}{(e^{10t} + e^{-10t})^2}, \]

This is clearly positive, so \( f(t) \) is always increasing. It is easy to see that the numerator is always smaller than the denominator, so this function must be between \(-1\) and \(1\).

b. To solve the integral, we make the substitution \( u = e^{10t} + e^{-10t} \), so \( du = 10(e^{10t} - e^{-10t}) dt \) with endpoints \( u = 2 \) for \( t = 0 \) and \( u = e^{10} + e^{-10} \) for \( t = 1 \). Thus,

\[ \int_{0}^{1} \frac{e^{10t} - e^{-10t}}{e^{10t} + e^{-10t}} dt = 0.1 \int_{2}^{e^{10} + e^{-10}} \frac{du}{u} = 0.1 \ln |u|_{u=2}^{e^{10} + e^{-10}} = 0.1(\ln(e^{10} + e^{-10}) - \ln(2)) = 0.1 \ln \left( \frac{e^{10} + e^{-10}}{2} \right) = 0.93. \]