1. The function, \( y = 2 - 4 \cos(2x) \), has a period of \( x = \pi \). The function oscillates about \( y = 2 \) with an amplitude of 4. It begins at a minimum at \((0, -2)\), goes to a maximum at \((\pi/2, 6)\), then ends its cycle at \((\pi, -2)\). The maxima occur at \( x = \pi/2, 3\pi/2 \). The graph of the function is below.

2. The function, \( y = 5 \sin(3x) - 4 \), has a period of \( x = 2\pi/3 \). The function oscillates about \( y = -4 \) with an amplitude of 5. It begins at \((0, -4)\), goes to a maximum at \((\pi/6, 1)\), continues through \((\pi/3, -4)\), then reaches a minimum at \((\pi/2, -9)\), and ends its cycle at \((2\pi/3, -4)\). The maxima occur at \( x = \pi/6, 5\pi/6, 3\pi/2 \). The graph of the function is above.

3. a. For the Logistic growth model, \( P_{n+1} = F(P_n) = 2.8P_n - 0.0005P_n^2 \) with \( P_0 = 1000 \), then

\[
P_1 = 2.8(1000) - 0.0005(1000)^2 = 2800 - 500 = 2300
\]
\[
P_2 = 2.8(2300) - 0.0005(2300)^2 = 3795
\]

b. At equilibrium, \( P_e = P_n = P_{n+1} \), so \( P_e = 2.8P_e - 0.0005P_e^2 \) or \( P_e(1.8 - 0.0005P_e) = 0 \). One solution is \( P_e = 0 \), and the other equilibrium satisfies \( 1.8 - 0.0005P_e = 0 \) or \( P_e = \frac{1.8}{0.0005} = 3600 \). The derivative of the updating function is \( F'(P) = 2.8 - 0.001P \). At \( P_e = 0 \), \( F'(0) = 2.8 > 1 \), so this equilibrium is unstable with solutions monotonically growing away from \( P_e = 0 \). At \( P_e = 3600 \), \( F'(3600) = 2.8 - 3.6 = -0.8 > -1 \), so the higher equilibrium is stable with solutions oscillating, but approaching \( P_e = 3600 \).

c. We see that \( F(P) = P(2.8 - 0.0005P) \), so the updating function has \( P \)-intercepts at \( P = 0 \) and \( P = 5600 \). The vertex has \( P_v = 2800 \), so \( F(2800) = 3920 \), which gives the vertex \((2800, 3920)\). The updating function intersects the identity function at the equilibria, \((0, 0)\) and \((3600, 3600)\). The graph is shown below.
4. a. For the population model with the Allee effect, \( N_{n+1} = N_n + 0.1N_n \left(1 - \frac{1}{9}(N_n - 5)^2\right) \) with (population in thousands) \( N_0 = 4 \), the next two generations are

\[
N_1 = 4 + 0.1(4) \left(1 - \frac{1}{9}(4 - 5)^2\right) = 4.356 \\
N_2 = 4.356 + 0.1(4.356) \left(1 - \frac{1}{9}(4.356 - 5)^2\right) = 4.771
\]

in thousands of birds.

b. \( N_e = N_e + 0.1N_e \left(1 - \frac{1}{9}(N_e - 5)^2\right) \), so \( 0.1N_e \left(1 - \frac{1}{9}(N_e - 5)^2\right) = 0 \). Thus, \( N_e = 0 \) or \( (N_e - 5)^2 = 9 \). It follows that the equilibria are \( N_e = 0, 2, \) and \( 8 \).

c. From the expanded model, \( N_{n+1} = A(N_n) = \frac{37}{40} N_n + \frac{1}{9} N_n^2 - \frac{1}{90} N_n^3 \), the derivative is \( A'(N) = \frac{37}{40} + \frac{2}{9} N - \frac{1}{90} N^2 \). At \( N_e = 0 \), \( A'(0) = \frac{37}{40} \), so this equilibrium is a stable equilibrium with solutions monotonically approaching \( 0 \). At \( N_e = 2 \), \( A'(2) = \frac{17}{15} \), so this equilibrium is an unstable equilibrium with solutions monotonically moving away from \( 2 \). At \( N_e = 8 \), \( A'(8) = \frac{7}{15} \), so this equilibrium is a stable equilibrium with solutions monotonically approaching \( 8 \).

d. Biologically, these results imply that if the population is below \( 2 \) thousand, then it will go to extinction (\( N_e = 0 \)). If the population is above \( 2 \) thousand, then the population of birds will grow to a carrying capacity of \( N_e = 8 \) thousand.

5. The area of the brochure is \( A = xy = 125 \), where \( x \) is the width of the page and \( y \) is the length of the page. The area of the printed page, which is to be maximized is given by

\[ P = (x-4)(y-5). \]

From the constraint on the page area, we have \( y = 125/x \), which when substituted above gives

\[ P(x) = (x-4) \left(\frac{125}{x} - 5\right) = 125 - \frac{500}{x} - 5x + 20 = 145 - 500x^{-1} - 5x \]

The maximum is found by differentiation, which gives

\[ P'(x) = 500x^{-2} - 5 = \frac{5(100 - x^2)}{x^2}. \]
This is zero when \( x = 10 \). It follows that \( y = 12.5 \). So the brochure has the dimensions 10×12.5 with the printed region having dimensions 6×7.5 or 45 in².

6. Combining the number of drops with the energy function, we have

\[
E(h) = hN(h) = h \left( 1 + \frac{10}{h - 1} \right) = h \left( \frac{h - 1 + 10}{h - 1} \right) = \frac{h^2 + 9h}{h - 1}.
\]

This is differentiated to give

\[
E'(h) = \frac{(h - 1)(2h + 9) - (h^2 + 9h)}{(h - 1)^2} = \frac{h^2 - 2h - 9}{(h - 1)^2}.
\]

A minimum occurs when \( h^2 - 2h - 9 = 0 \), so

\[
h = 1 \pm \sqrt{10} = -2.1623, 4.1623.
\]

It follows that the minimum energy occurs when \( h = 1 + \sqrt{10} = 4.1623 \) m, which gives the height that a crow should fly to minimize the energy needed to break open a walnut.

7. a. If \( P_0 = 100 \), then \( P_1 = 600e^{-0.1} = 542.9 \) and \( P_2 = 6(542.9)e^{-0.5429} = 1892.75 \).

b. The derivative is \( R'(P) = 6e^{-0.001P}(1 - 0.001P) \). The critical \( P_c \) occurs at \( P_c = 1000 \), so there is a maximum at \((1000, 6000e^{-1}) = (1000, 2207)\). The graph passes through the origin, so \((0, 0)\) is the only intercept. Since \( \lim_{P \to \infty} R(P) = 0 \), the is a horizontal asymptote at \( R = 0 \). The second derivative is \( R''(P) = -0.006e^{-0.001P}(2 - 0.001P) \), which is zero at \( P = 2000 \). Thus, there is a point of inflection at \((2000, 12000e^{-2}) = (2000, 1624)\). The graph is below.

c. The equilibria satisfy \( P_e = 6P_e^{-0.001P_e} \), so either \( P_e = 0 \) or \( 1 = 6e^{-0.001P_e} \). The latter gives \( P_e = 1000 \ln(6) \approx 1792 \). For \( P_e = 0 \), \( R'(0) = 6 > 1 \), so this equilibrium is unstable with solutions moving monotonically away. For \( P_e = 1792 \), \( R'(1792) = -0.7918 \), so this equilibrium is stable with solutions oscillating and moving toward the equilibrium.

![Problem 7](image1.png)

![Problem 8](image2.png)
8. a. If \( P_0 = 500 \), then \( P_1 = \frac{8000}{3.5^2} = 653.06 \) and \( P_2 = 574.346 \).

b. The derivative is \( H'(P) = \frac{16(1 + 0.005P)^2 - 32P(1 + 0.005P)(0.005)}{(1 + 0.005P)^4} = \frac{16(1 - 0.005P)}{(1 + 0.005P)^3} \).

The critical \( P_c \) occurs at \( P_c = 200 \), so there is a maximum at \((200, 800)\). The graph passes through the origin, so \((0,0)\) is the only intercept. Since \( \lim_{P \to \infty} H(P) = 0 \), the is a horizontal asymptote at \( H = 0 \). The second derivative is
\[
H''(P) = \frac{-0.08(1 + 0.005P)^3 - 0.24(1 - 0.005P)(1 + 0.005P)^2}{(1 + 0.005P)^6} = \frac{-0.16(2 - 0.005P)}{(1 + 0.005P)^4},
\]
which is zero at \( P = 400 \). Thus, there is a point of inflection at \((400, 6400/9) = (400, 711)\). The graph is above.

c. The equilibria satisfy \( P_e = \frac{16P_e}{(1 + 0.005P_e)^2} \), so either \( P_e = 0 \) or \((1 + 0.005P_e)^2 = 16 \). The latter gives \( P_e = 600 \) (neglecting the negative solution). For \( P_e = 0 \), \( H'(0) = 16 > 1 \), so this equilibrium is unstable with solutions moving monotonically away. For \( P_e = 600 \), \( H'(600) = -0.5 \), so this equilibrium is stable with solutions oscillating and moving toward the equilibrium.

9. a. At rest, \( V(t) = -70 = 50(t - 2)(t - 3) - 70 \), so \( 50(t - 2)(t - 3) = 0 \). Thus, the membrane is at rest when \( t = 0, 2, \) and \( 3 \).

b. To find the extrema, we first write \( V(t) = 50(t^3 - 5t^2 + 6t) - 70 \), then the derivative is \( V'(t) = 50(3t^2 - 10t + 6) \). By the quadratic formula, \( t = \frac{5 \pm \sqrt{7}}{3} = 0.7847, 2.5486 \). Substituting these values into the membrane equation gives the peak of the action potential at \( t = 0.7847 \) with a membrane potential of \( V(0.7847) = 35.63 \text{ mV} \), while the minimum potential (most hyperpolarized state) occurs at \( t = 2.5486 \) with a membrane potential of \( V(2.5486) = -101.56 \text{ mV} \). Below is a graph for this model of membrane potential.

10. a. The time as a function of \( x \) is given by
\[
T(x) = \frac{50 - x}{15} + \frac{(x^2 + 1600)^{1/2}}{9}.
\]
b. We differentiate $T(x)$ to find the minimum time,

$$T'(x) = -\frac{1}{15} + \frac{1}{9} \left( \frac{1}{2} (x^2 + 1600)^{-1/2} \right) = -\frac{1}{15} + \frac{x}{9(x^2 + 1600)^{1/2}}.$$ 

Setting this derivative equal to zero gives

$$\frac{x}{9(x^2 + 1600)^{1/2}} = \frac{1}{15}$$

$$5x = 3(x^2 + 1600)^{1/2}$$

$$25x^2 = 9(x^2 + 1600)$$

$$16x^2 = 14400$$

$$x^2 = 900$$

This implies $x = 30$ m produces the minimum time. $T(30) = \frac{20}{15} + \frac{50}{9} = \frac{62}{9} = 6.89$ sec. We check the endpoints $T(0) = \frac{70}{9} = 7.778$ sec and $T(50) = \frac{100\sqrt{41}}{9} = 7.11$ sec, confirming the optimal escape strategy is for the rabbit to run 20 m along the road, then run straight toward the burrow.

11. The **objective function** is given by:

$$S(x, y) = 2x^2 + 7xy.$$ 

The constraint condition is given by:

$$V = x^2 y = 50,000 \text{ cm}^3, \text{ so } y = \frac{50,000}{x^2}.$$ 

Thus,

$$S(x) = 2x^2 + \frac{350,000}{x}.$$ 

Differentiating we have,

$$S'(x) = 4x - \frac{350,000}{x^2}.$$ 

Solving $S'(x) = 0$, so $x^3 = \frac{350,000}{4} = 87,500$ or $x = 44.395$. It follows $y = 25.37$. Thus, the minimum amount of material needed is $S(44.395) = 11,825.6 \text{ cm}^2$.

12. From the diagrams, we have that $r^2 + h^2 = a^2$, which gives $h^2 = a^2 - r^2$. The circumference of the base of the cone is $2\pi r = a\theta$, where $\theta$ is in radians. (Radians are an easy means of determining the length of a sector of a circle.) Thus, $r = a\theta/2\pi$. It follows that $h^2 = a^2 - a^2\theta^2/(4\pi^2)$. The volume of the water cup is given by

$$V = \frac{\pi r^2 h}{3} = \frac{\pi}{3} \left( \frac{a\theta}{2\pi} \right)^2 \sqrt{a^2 - \frac{a^2\theta^2}{4\pi^2}}.$$ 

$$V(\theta) = \frac{a^3\theta^2}{12\pi} \sqrt{1 - \frac{\theta^2}{4\pi^2}} = \frac{a^3}{12\pi} \theta^2 \left( 1 - \frac{\theta^2}{4\pi^2} \right)^{1/2}.$$ 

This expression is differentiated with respect to $\theta$.

$$V'(\theta) = \frac{a^3}{12\pi} \left( \frac{\theta^2}{2} \left( 1 - \frac{\theta^2}{4\pi^2} \right)^{-1/2} \left( -\frac{2\theta}{4\pi^2} \right) + 2\theta \left( 1 - \frac{\theta^2}{4\pi^2} \right)^{1/2} \right)$$.
\[
\frac{a^3}{12\pi \left(1 - \frac{\theta^2}{4\pi^2}\right)^{1/2}} \left(-\frac{\theta^3}{4\pi^2} + 2\theta \left(1 - \frac{\theta^2}{4\pi^2}\right)\right)
\]

\[
\frac{a^3\theta}{12\pi \left(1 - \frac{\theta^2}{4\pi^2}\right)^{1/2}} \left(2 - 3\frac{\theta^2}{4\pi^2}\right)
\]

The maximum is found by setting this derivative above equal to zero, so \(2 - \frac{3\theta^2}{4\pi^2} = 0\). It follows that \(\theta^2 = \frac{8\pi^2}{3}\) or

\[\theta = 2\pi \sqrt{\frac{2}{3}} \simeq 5.1302.\]

Thus, \(\theta = 2\pi \sqrt{\frac{2}{3}} \simeq 5.1302\) radians (which is about 294°), so a sector of 1.1530 radians or about 66° is removed. The dimensions of the cone should have a radius of \(r = a\sqrt{\frac{2}{3}} \simeq 0.8165a\) and a height of \(h = a\sqrt{\frac{1}{3}} \simeq 0.57735a\).

13. a. \(L(0) = 0.24\) m (24 cm) is the birth size a leopard shark (\(L\)-intercept). For large \(t\), \(L(t) \to 1.6\) m. The graph of this von Bertalanffy equation is shown below. Sexual maturity is found by solving \(L(t) = 0.5 = 1.6(1 - 0.85e^{-0.08t})\) or \(1.36e^{-0.08t} = 1.1\) or \(e^{0.08t} = 1.236\). It follows that sexual maturity occurs at \(t = 2.652\) yr.

b. The composite function is given by

\[W(t) = 4.5(1.6(1 - 0.85e^{-0.08t}))^3 = 18.432(1 - 0.85e^{-0.08t})^3.\]

The intercept is \(W(0) = 0.0622\) kg, while for large \(t\), \(W(t) \to 18.432\) kg. The graph of this function is shown below.

c. By the chain rule, the derivative of \(W(t)\) is

\[W'(t) = 3(18.432)(1 - 0.85e^{-0.08t})^2(-0.85)(-0.08)e^{-0.08t} = 3.76e^{-0.08t}(1 - 0.85e^{-0.08t})^2.\]

By the product rule and chain rule, the second derivative is

\[W''(t) = 3.76 \left(2e^{-0.08t}(1 - 0.85e^{-0.08t})(-0.85)(-0.08)e^{-0.08t} - 0.08e^{-0.08t}(1 - 0.85e^{-0.08t})^2\right)\]

\[W''(t) = 3.76e^{-0.08t}(1 - 0.85e^{-0.08t})(0.204e^{-0.08t} - 0.08)\]

\(W''(t) = 0\) when either \(1 - 0.85e^{-0.08t} = 0\) or \(0.204e^{-0.08t} - 0.08 = 0\). The first is zero when \(t = -2.03\) yr, while the second is zero when \(t = 11.7\) yr. It follows that the maximum weight gain occurs at age \(t = 11.7\) yr with a weight gain of \(W'(11.7) = 0.655\) kg/yr.

14. a. The periodic contractions of 10/min implies that \(0.1\omega = 2\pi\) or \(\omega = 20\pi\). The average value \(A = \frac{1 + 1}{2} = 2.5\), while the amplitude is given by \(B = 4 - 2.5 = 1.5\). Thus, the radius of the small intestine is given by

\[R(t) = 2.5 + 1.5\cos(20\pi t).\]

b. The graph of \(R(t)\) for \(t \in [0, 0.2]\) is shown below. The maxima occur at \(t = 0, 0.1, 0.2\) min, and the minima are halfway between the maxima with \(t = 0.05, 0.15\) min.
15. The period is 365 days, so \(365\omega = 2\pi\) or \(\omega = \frac{2\pi}{365} \approx 0.01721\). The average length of time is \(\alpha = \frac{1162+327}{2} = 744.5\) min. The amplitude is given by \(\beta = 1162 - 744.5 = 417.5\) min. The maximum occurs on day 170, so \(\omega(170 - \phi) = \pi/2\) (based on the maximum of the sine function). Thus, \(170 - \phi = \frac{365}{4} = 91.25\) or \(\phi = 78.75\) day. It follows that

\[
L(t) = 744.5 + 417.5\sin(0.01721(t - 78.75)).
\]

The length of day for Ground Hog’s day is \(L(32) = 744.5 + 417.5\sin(0.01721(32 - 78.75)) = 443.7\) min in Anchorage.

16. a. From \(P_3\), we have \(P_3 = 68.34 = 28.49(1+r)^3\), so \((1+r) = (68.34/28.49)^{1/3} = 1.33863\). Thus, \(r = 0.33863\). Doubling time satisfies \(2P_0 = P_0(1+r)^n\) or \(n = \ln(2)/\ln(1+r) = 2.377\) decades or 23.77 years.

b. The model predicts the population in 2000 is \(P_5 = 28.49(1.33863)^5 = 122.46\) million. The percent error is \(100\frac{(122.46 - 99.93)}{99.93} = 22.55\%\).

c. From the logistic model, we obtain \(P_1 = 39.32\) million and \(P_2 = 52.79\) million.
d. To find equilibria, we solve \( P_e = 1.48P_e - 0.0035P_e^2 \), which gives \( P_e = 0 \) or \( P_e = 137.14 \) million. The derivative of the updating function is \( F'(P) = 1.48 - 0.007P_e \), so \( F'(137.14) = 0.52 \). It follows that this equilibrium is stable with solutions monotonically approaching this carrying capacity equilibrium.

17. From the high and low temperatures, \( A \) is the average, so \( A = 18^\circ \text{C} \). The amplitude \( B \) is the difference between the maximum and the average, so \( B = 8^\circ \text{C} \). The period is 24 hr, so \( 24\omega = 2\pi \) or \( \omega = \frac{\pi}{12} \approx 0.2618 \). The minimum temperature occurs at 4 AM \((t = 4)\), so

\[
T(4) = 10 = 18 - 8\sin\left(\frac{\pi}{12}(4 - \phi)\right).
\]

It follows that

\[
\sin\left(\frac{\pi}{12}(4 - \phi)\right) = 1 \quad \text{or} \quad \frac{\pi}{12}(4 - \phi) = \frac{\pi}{2}.
\]

Hence, \( \phi = -2 \). If we want \( \phi \in [0, 24] \), then by periodicity we can simply add 24 to obtain \( \phi = 22 \). (Both answers for \( \phi \) are correct, but if the restriction on \( \phi \) is required, we can only obtain the second answer.)