2. a. Solve

$$\frac{dz}{dt} = 0.1z - 2, \quad z(0) = 5.$$

Factor the coefficient leading $z(t)$ to give

$$z'(t) = 0.1(z(t) - 20).$$

Make the substitution, $w(t) = z(t) - 20$, so $w(0) = z(0) - 20 = -15$ and $w'(t) = z'(t)$. This gives the problem

$$w' = 0.1w, \qquad w(0) = -15.$$

This has the solution

$$w(t) = -15e^{0.1t} = z(t) - 20, \quad \text{so} \quad z(t) = 20 - 15e^{0.1t}.$$

b. Solve

$$\frac{dh}{dx} = 5 - 0.2h, \quad h(0) = 10.$$

Factor the coefficient leading $h(x)$ to give

$$h'(x) = -0.2(h(x) - 25).$$

Make the substitution, $z(x) = h(x) - 25$, so $z(0) = h(0) - 25 = -15$ and $z'(x) = h'(x)$. This gives the problem

$$z' = -0.2z, \qquad z(0) = -15.$$

This has the solution

$$z(x) = -15e^{-0.2x} = h(x) - 25, \quad \text{so} \quad h(x) = 25 - 15e^{-0.2x}.$$

4. Solve

$$\frac{dy}{dt} = 0.02y, \quad y(2) = 50.$$

This is like Malthusian growth except that the initial condition begins at $t = 2$. The general solution is

$$y(t) = y_0e^{0.02t} \quad \text{with} \quad y(2) = 50 = y_0e^{0.04}.$$

Thus, $y_0 = 50e^{-0.04}$. This gives the solution

$$y(t) = 50e^{-0.04}e^{0.02t} = 50e^{0.02(t-2)}.$$

7. a. The differential equation for the population of Canada is

$$C'(t) = k_1C(t), \quad C(0) = 24,070,000,$$

which by letting $t = 0$ corresponds to 1980 has the solution

$$C(t) = 24,070,000e^{k_1 t}.$$

From the population in 1990, we have

$$C(10) = 26,620,000 = 24,070,000e^{10k_1} \quad \text{or} \quad e^{10k_1} = \frac{26,620,000}{24.070,000}.$$

It follows that $k_1 = \frac{1}{10}\ln(26620/24070) \simeq 0.01007 \text{ yr}^{-1}$. To find doubling time, we solve

$$C(t_d) = 2(24,070,000) = 24,070,000e^{k_1 t_d} \quad \text{or} \quad e^{k_1 t_d} = 2.$$

It follows that the doubling time, $t_d = \frac{1}{k_1}\ln(2) \simeq 68.8$ yr.

b. A similar argument gives the population of Kenya, $K(t) = 16,681,000e^{k_2 t}$, with the rate constant $k_2 = \frac{1}{10}\ln(24229/16681) \simeq 0.03733 \text{ yr}^{-1}$ and doubling time, $t_d = \frac{1}{k_2}\ln(2) \simeq 18.6$ yr.

c. The populations in 2000 are given by $C(20) = 24,070,000e^{20k_1} = 29,440,150$ and $K(20) = 16,681,000e^{20k_2} = 35,192,401$. The populations are equal when $C(t) = K(t)$ or

$$24,070,000e^{k_1 t} = 16,681,000e^{k_2 t} \quad \text{or} \quad \frac{e^{k_2 t}}{e^{k_1 t}} = \frac{24,070}{16,681}.$$

Thus, $e^{(k_2 - k_1)t} = \frac{24,070}{16,681}$, so $(k_2 - k_1)t = \ln\left(\frac{24,070}{16,681}\right)$. It follows that

$$t = \frac{1}{k_2 - k_1}\ln\left(\frac{24,070}{16,681}\right) \simeq 13.5 \text{ years.}$$

This would be in the middle of 1993.

A. a. The radioactive decay problem is

$$R' = -kR, \qquad R(0) = 20.$$

This has the solution $S(t) = 20e^{-kt}$. Since the half-life is 28 years, we have

$$S(28) = 10 = 20e^{-28k}, \quad \text{so} \quad e^{28k} = 2.$$

Thus, $28k = \ln(2)$ or $k = \frac{1}{28}\ln(2) \simeq 0.02476 \text{ yr}^{-1}$. After 10 years, $S(10) = 20e^{-10k} = 15.6$ mg.

b. For 7 mg remaining, $S(t) = 20e^{-kt} = 7$ or $e^{kt} = 20/7$. It readily follows that $t = \ln(20/7)/k \simeq 42.4$ yr.

C. a. From the notes, we have $C_a = QT/(P_{sys} - P_{dia})$ and $P_{dia} = P_{sys}e^{-T/C_a R_S}$. Thus, $P_{sys} - P_{dia} = P_{sys}\left(1 - e^{-T/C_a R_S}\right) = QT/C_a$, so

$$P_{sys} = \frac{QT}{C_a\left(1 - e^{-T/C_a R_S}\right)}.$$

By letting $Q = 5.6$ liters/min, $T = 1/70$ min/beats, $C_a = 0.0016$ liters/mm Hg, and $R_s = 17.6$ (mm Hg/liter/min) and substituting into the formula above, we find the systolic pressure as $P_{sys} = 125.7.1$ mm Hg. It follows with the formula for diastolic pressure that $P_{dia} =$ 75.6 mm Hg, respectively. Thus, the 20% decrease in compliance causes minimal changes in the blood pressure.

10. a. The differential equation is given by $T'(t) = -k(T(t) - 20), \quad T(0) = 100$. Let $z(t) = T(t) - 20$, then $z(0) = 100 - 20 = 80$ and $z'(t) = T'(t)$. Thus, we need to solve the initial value problem

$$z'(t) = -kz(t), \qquad z(0) = 80.$$

This has the solution $z(t) = 80e^{-kt} = T(t) - 20$, so $T(t) = 20 + 80e^{-kt}$.

b. Since $T(10) = 80 = 20 + 80e^{-10k}$, $60 = 80e^{-10k}$ or $e^{10k} = 80/60 = 4/3$. It follows that the rate constant $k = \ln(4/3)/10 \simeq 0.02877$. To find when it reaches 30°C, we solve $T(t) = 30 = 20 + 80e^{-kt}$ or $10 = 80e^{-kt}$ or $e^{kt} = 8$. This gives $t = \ln(8)/k \simeq 72.3$ min.

12. a. Let $a(t)$ be the amount of pollutant, and the concentration $c(t)$ is the concentration of pollutant (in ppb). The change in amount = the amount entering - the amount leaving. The change in amount, $a'(t)$, has units (mass/day). The amount entering is $fQ = 20,000$ ppb m^3/day (mass/day), while the amount leaving is $fc(t) = 4000c(t)$ ppb m^3/day (mass/day). Thus,

$$a'(t) = 20000 - 4000c(t).$$

To make the concentration equation, we use that $c(t) = a(t)/V = a(t)/200000$. Since $c'(t) = a'(t)/200000$, we can write the equation above as

$$c'(t) = 0.1 - 0.02c(t) \quad \text{with} \quad c(0) = 0.$$

To solve this initial value problem we write

$$c' = -0.02(c - 5) \quad \text{with} \quad c(0) = 0.$$

We make the substitution $z(t) = c(t) - 5$, so $z(0) = c(0) - 5 = -5$ and $z'(t) = c'(t)$. The concentration equation becomes

$$z' = -0.02z \quad \text{with} \quad z(0) = -5,$$

which has the solution

$$z(t) = -5e^{-0.02t} = c(t) - 5.$$

Thus, the solution is $c(t) = 5 - 5e^{-0.02t}$.

b. We solve $c(t) = 5 - 5e^{-0.02t} = 4$ or $5e^{-0.02t} = 1$. Thus, $e^{0.02t} = 5$ or $0.02t = \ln(5)$. Hence, the concentration reaches 4 ppb at $t = 50\ln(5) \simeq 80.5$ days.

c. The limiting concentration is $\lim_{t\to\infty} c(t) = 5$ ppb. This easily follows because as $t \to \infty$, $e^{-0.02t} \to 0$.

13. a. Let $a(t)$ be the amount of pollutant, and the concentration $c(t)$ is the concentration of pollutant (in ppb). The change in amount = the amount entering - the amount leaving. The change in amount, $a'(t)$, has units (mass/day). The amount entering is $f_1Q_1 + f_2Q_2 =$

$4000\cdot 18+2500\cdot 4 = 82,000$ ppb m^3/day (mass/day), while the amount leaving is $(f_1+f_2)c(t) = 6500c(t)$ ppb m^3/day (mass/day). Thus, the differential equation for the change in amount is

$$\frac{da(t)}{dt} = 82,000 - 6500c(t).$$

To make the concentration equation, we use that $c(t) = a(t)/V = a(t)/3000000$. Since $c'(t) = a'(t)/3000000$, we can write the equation above as

$$\frac{dc(t)}{dt} = \frac{82,000}{3,000,000} - \frac{6500}{3,000,000}c = -\frac{13}{6000}\left(c - \frac{164}{13}\right) = -0.0021667(c - 12.615).$$

The initial condition is $c(0) = 0$. To solve this initial value problem, we make the substitution $z(t) = c(t) - 12.615$, so $z(0) = c(0) - 12.615 = -12.615$ and $z'(t) = c'(t)$. The concentration equation becomes

$$\frac{dz}{dt} = -0.0021667z \quad \text{with} \quad z(0) = -12.615,$$

which has the solution

$$z(t) = -12.615e^{-0.0021667t} = c(t) - 12.615.$$

Thus, the solution is $c(t) = 12.615(1 - e^{-0.0021667t})$.

b. We solve $c(t) = 12.615(1 - e^{-0.0021667t}) = 4$ or $12.615e^{-0.0021667t} = 8.615$. Thus, $e^{0.0021667t} = \frac{12.615}{8.615} = 1.4643$ or $0.0021667t = \ln(1.4643)$. Hence, the concentration reaches 4 ppb at $t \approx 176.01$ days.

The limiting concentration is $\lim_{t\to\infty} c(t) = 12.615$ ppb. This easily follows because as $t \to \infty$, $e^{-0.0021667t} \to 0$.