1. Solutions for the integrals:

b. For the integral below, we make the substitution $u = 2x^2 - 3$, so $du = 4x\,dx$. It follows that:

$$\begin{aligned}
\int x\sqrt{2x^2-3}\,dx &= \frac{1}{4}\int \left(2x^2-3\right)^{\frac{1}{2}}(4x)dx \\
&= \frac{1}{4}\int u^{\frac{1}{2}}du \\
&= \frac{1}{6}u^{\frac{3}{2}} + C \\
&= \frac{1}{6}(2x^2-3)^{\frac{3}{2}} + C
\end{aligned}$$

d. For the integral below, we make the substitution $u = x^2 + 4x - 5$, so $du = (2x+4)\,dx$. It follows that:

$$\begin{aligned}
\int \frac{x+2}{(x^2+4x-5)^3}\,dx &= \frac{1}{2}\int \left(x^2+4x-5\right)^{-3}(2x+4)dx \\
&= \frac{1}{2}\int u^{-3}du \\
&= -\frac{1}{4}u^{-2} + C \\
&= -\frac{1}{4\left(x^2+4x-5\right)^2} + C
\end{aligned}$$

f. For the integral below, we make the substitution $u = -(x^2 - 2x)$, so $du = -(2x-2)\,dx$. It follows that:

$$\begin{aligned}
\int \frac{x-1}{e^{x^2-2x}}\,dx &= -\frac{1}{2}\int e^{-(x^2-2x)}(-2(x-1))dx \\
&= -\frac{1}{2}\int e^u\,du \\
&= -\frac{1}{2}e^u + C \\
&= -\frac{1}{2}e^{-x^2+2x} + C
\end{aligned}$$

h. For the integral below, we make the substitution $u = \cos(2x)$, so $du = -2\sin(2x)\,dx$. It follows that:

$$\begin{aligned}
\int \frac{\sin(2x)}{\cos(2x)}\,dx &= -\frac{1}{2}\int \frac{1}{\cos(2x)}(-2\sin(2x))\,dx \\
&= -\frac{1}{2}\int \frac{1}{u}\,du \\
&= -\frac{1}{2}\ln|u| + C \\
&= -\frac{1}{2}\ln|\cos(2x)| + C
\end{aligned}$$

,

i. For the integral below, we make the substitution $u = x^{\frac{1}{2}}$, so $du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$. It follows that:

$$\begin{aligned}
\int \frac{e^{\sqrt{x}}}{\sqrt{x}}, dx &= 2\int e^{\sqrt{x}}\frac{1}{2}x^{-\frac{1}{2}}\,dx \\
&= 2\int e^{u}\,du \\
&= 2\,e^{u} + C \\
&= 2\,e^{\sqrt{x}} + C
\end{aligned}$$

2. Solutions for the differential equations.

a. The differential equation below has only a function of t on the right, so it can be solved by simply integrating the right hand side.

$$\begin{aligned}
\frac{dy}{dt} &= t\sqrt{t^2+1} \\
y(t) &= \int t\sqrt{t^2+1}\,dt
\end{aligned}$$

The integral is solved with the substitution $u = t^2 + 1$, so $du = 2t\,dt$.

$$\begin{aligned}
\int t\sqrt{t^2+1}\,dt &= \frac{1}{2}\int (t^2+1)^{\frac{1}{2}}2t\,dt \\
&= \frac{1}{2}\int u^{\frac{1}{2}}du \\
&= \frac{1}{3}u^{\frac{3}{2}} + C \\
&= \frac{1}{3}(t^2+1)^{\frac{3}{2}} + C
\end{aligned}$$

Thus,

$$y(t) = \frac{1}{3}(t^2+1)^{\frac{3}{2}} + C.$$

From the initial condition, $y(0) = 5$, we have $5 = \frac{1}{3} + C$ or $C = \frac{14}{3}$. Thus, the solution is given by

$$y(t) = \frac{1}{3}\left(t^2+1\right)^{3/2} + \frac{14}{3}.$$

d. The differential equation below is a separable differential equation that has already been separated.

$$y\frac{dy}{dt} = \frac{1-t}{(t^2-2t+2)^3}.$$

The integral in t uses the substitution $u = t^2 - 2t + 2$, so $du = (2t-2)\,dt$, so from the separable variables techniques

$$\int y\,dy = \int \frac{1-t}{(t^2-2t+2)^3}\,dt$$

$$
\begin{aligned}
\frac{1}{2}y^2(t) &= -\frac{1}{2}\int (t^2-2t+2)^{-3}(2(t-1))\,dt \\
y^2(t) &= -\int u^{-3}\,du \\
y^2(t) &= \frac{1}{2}u^{-2}+C \\
y^2(t) &= \frac{1}{2(t^2-2t+2)^2}+C
\end{aligned}
$$

Thus,

$$
y(t) = \pm\sqrt{\frac{1}{2(t^2-2t+2)^2}+C}.
$$

From the initial condition, $y(1) = 1$, we have $1 = \sqrt{\frac{1}{2}+C}$ or $C = \frac{1}{2}$. Thus, the solution is given by

$$
y(t) = \sqrt{\frac{1}{2(t^2-2t+2)^2}+\frac{1}{2}}.
$$

e. The differential equation below is a separable differential equation, which gives the following two integrals.

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{t^2}{y}e^{-y^2} \\
\int y\,e^{y^2}\,dy &= \int t^2 dt
\end{aligned}
$$

The integral in y uses the substitution $u = y^2$, so $du = 2y\,dy$, so

$$
\begin{aligned}
\frac{1}{2}\int e^{y^2}2y\,dy &= \int t^2 dt \\
\frac{1}{2}\int e^u du &= \frac{1}{3}t^3+C \\
\frac{1}{2}e^u &= \frac{1}{3}t^3+C \\
e^{y^2} &= \frac{2}{3}t^3+C_1 \\
y^2(t) &= \ln\left|\frac{2}{3}t^3+C_1\right|
\end{aligned}
$$

Thus,

$$
y(t) = \pm\sqrt{\ln\left|\frac{2}{3}t^3+C_1\right|}.
$$

From the initial condition, $y(0) = 0$, we have $0 = \sqrt{\ln(C_1)}$ or $C_1 = 1$. Thus, the solution is given by

$$
y(t) = \sqrt{\ln\left(\frac{2}{3}t^3+1\right)}.
$$

g. The differential equation below is a separable differential equation, which gives the following two integrals.

$$\begin{aligned} t\frac{dy}{dt} &= (\ln(t))^2 \\ \int dy &= \int (\ln(t))^2 \frac{1}{t}\, dt \end{aligned}$$

The integral in t uses the substitution $u = \ln(t)$, so $du = \frac{1}{t}\, dt$, so

$$\begin{aligned} y(t) &= \int u^2 dt \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(\ln(t))^3 + C \end{aligned}$$

Thus,

$$y(t) = \frac{1}{3}(\ln(t))^3 + C.$$

From the initial condition, $y(1) = -2$, we have $-2 = \frac{1}{3}(\ln(1))^3 + C$ or $C = -2$. Thus, the solution is given by

$$y(t) = \frac{1}{3}(\ln(t))^3 - 2.$$

3. From the lecture notes, the equation governing the motion of an object subject to gravity is given by

$$m\frac{dv}{dt} = -\frac{mgR^2}{(x+R)^2},$$

which when using the relation that $\frac{dv}{dt} = v\frac{dv}{dx}$ gives

$$v\frac{dv}{dx} = -\frac{gR^2}{(x+R)^2}.$$

Using separation of variables, we see

$$\int v\, dv = -gR^2 \int \frac{dx}{(x+R)^2}.$$

Thus,

$$\frac{v^2(x)}{2} = \frac{gR^2}{x+R} + C.$$

With the initial condition, $v(R) = V_0$, we can readily solve for the constant and $2C = V_0^2 - 2gR$, so

$$v(x) = \sqrt{\frac{2gR^2}{x+R} + V_0^2 - 2gR} = \sqrt{\frac{797,306}{x+6378} - 100.0088}$$

with the data that $R = 6378$ km, $g = 0.0098$ km/sec^2, and $V_0 = 5$ km/sec. The velocity is zero at the maximum height, so we solve

$$0 = \sqrt{\frac{797,306}{x + 6378} - 100.0088}.$$

This equation gives the maximum height achieved by the object is 1594 km.

5. a. This problem satisfies the properties of logistic growth, and it can be rewritten in the following form:

$$\frac{dP}{dt} = -0.3P\left(\frac{P}{300} - 1\right),$$

which we saw in the lecture notes could be separated into the following integral form:

$$\frac{1}{300}\int \frac{1}{\frac{P}{300} - 1} dP - \int dP = -0.3\int dt.$$

Following the notes, these integrals are evaluated to give

$$\ln\left|\frac{P(t)}{300} - 1\right| - \ln|P(t)| = -0.3t + C.$$

If $A = e^C$, then some algebra gives the general solution

$$P(t) = \frac{300}{1 - 300Ae^{0.3t}}.$$

With the initial condition that $P(0) = 1$, the population of game fish is given by

$$P(t) = \frac{300}{1 + 299e^{-0.3t}}.$$

From the form of the equation above, it is easy to see that the carrying capacity for the game fish is 300 (thousand). 90% of the carrying capacity is 270 (thousand), so we solve

$$270 = \frac{300}{1 + 299e^{-0.3t}} \qquad \text{or} \qquad 1 + 299e^{-0.3t} = \frac{10}{9}.$$

Thus, $e^{0.3t} = 9(299) = 2691$ or $t = \frac{10}{3}\ln(2691) = 26.3$ yr to reach 90% of this carrying capacity.

b. The differential equation

$$\frac{dP}{dt} = -0.001(P^2 - 300P + 20,000)$$

can be readily separated to give

$$\int \frac{dP}{P^2 - 300P + 20,000} = -0.001\int dt.$$

From the formula given

$$\begin{aligned}
\int \left(\frac{0.01}{P-200} - \frac{0.01}{P-100} \right) dP &= -0.001t + C \\
0.01 \ln|P(t)-200| - 0.01 \ln|P(t)-100| &= -0.001t + C \\
\ln \left| \frac{P(t)-200}{P(t)-100} \right| &= -0.1t + C_1 \\
\frac{P(t)-200}{P(t)-100} &= e^{-0.1t+C_1} = Ae^{-0.01t}
\end{aligned}$$

With the initial condition, $P(0) = 300$, $A = \frac{1}{2}$. Thus, implicitly

$$\frac{P(t)-200}{P(t)-100} = \frac{1}{2} e^{-0.01t}.$$

This is solved to give the solution to the differential equation with harvesting

$$P(t) = \frac{400 - 100e^{-0.1t}}{2 - e^{-0.1t}}.$$

c. By letting $t \to \infty$ in the equation above, one easily finds that the limiting population for this differential equation with harvesting is 200 (thousand) game fish. The graph can be found in the short solutions.

7. a. To find the maximum growth rate, we differentiate $k(t)$, so by the quotient rule

$$k'(t) = 0.12 \frac{(t^2+1) \cdot 1 - t(2t)}{(t^2+1)^2} = 0.12 \frac{1-t^2}{(t^2+1)^2}.$$

Setting the derivative equal to zero gives $t = \pm 1$, and $k(1) = \frac{0.12}{2} = 0.06$. Thus, the maximum growth rate, $k(t)$, is 0.06, occurring at $t = 1$. The graph of this function is available on the short solutions.

b. The differential equation given by

$$\frac{dV}{dt} = 0.12 \frac{t}{t^2+1} V^{2/3}$$

is a separable equation that can be solved from the integrals

$$\begin{aligned}
\int V^{-2/3}\, dV &= 0.12 \int (t^2+1)^{-1} t\, dt \\
3\, V^{1/3} &= 0.06 \int (t^2+1)^{-1} (2t)\, dt \\
V^{1/3}(t) &= 0.02 \int u^{-1} du \\
V^{1/3}(t) &= 0.02 \ln|u| + C = 0.02 \ln(t^2+1) + C
\end{aligned}$$

where we took $u = t^2 + 1$ and $du = 2t\,dt$. Since $V(0) = 1$,

$$1 = 0.02\ln(1) + C \qquad \text{or} \qquad C = 1.$$

By cubing both sides we see that the solution to the differential equation is

$$V(t) = \left(1 + 0.02\ln(t^2 + 1)\right)^3.$$

c. To double its size the cell must satisfy $2 = \left(1 + 0.02\ln(t^2 + 1)\right)^3$ or $1+0.02\ln(t^2+1) = 2^{1/3}$. Thus, $\ln(t^2 + 1) = 50\left(2^{1/3} - 1\right)$ or $t^2 = \exp\left(50\left(2^{1/3} - 1\right)\right) - 1$. By taking the square root, we find that it takes 664 min for the cell to double in volume.