1. Evaluate the following integrals:

a. $\int (2-x)^2\, dx,$

b. $\int x\sqrt{2x^2-3}\, dx,$

c. $\int \frac{4}{2x-6}\, dx,$

d. $\int \frac{x+2}{(x^2+4x-5)^3}\, dx,$

e. $\int xe^{1-x^2}\, dx,$

f. $\int \frac{x-1}{e^{x^2-2x}}\, dx,$

g. $\int x\ \sin(x^2+4), dx,$

h. $\int \frac{\sin(2x)}{\cos(2x)}\, dx,$

i. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}, dx,$

j. $\int \frac{x^3}{\sqrt{1-x^4}}\, dx,$

2. Solve the following differential equations.

a. $\frac{dy}{dt} = t\sqrt{t^2+1}, \quad y(0)=5,$

b. $\frac{dy}{dt} = t\sin(t^2-4) \quad y(2)=3,$

c. $\frac{dy}{dt} = \frac{ty}{\sqrt{t^2-1}}, \quad y(1)=4,$

d. $y\frac{dy}{dt} = \frac{1-t}{(t^2-2t+2)^3}, \quad y(1)=1,$

e. $\frac{dy}{dt} = \frac{t^2}{y}e^{-y^2}, \quad y(0)=0,$

f. $\frac{dy}{dt} = 0.1t(4-y), \quad y(0)=10,$

g. $t\frac{dy}{dt} = (\ln(t))^2, \quad y(1)=-2,$

h. $\frac{dy}{dt} = 0.2y\left(1-\frac{y}{10}\right), \quad y(0)=2.$

3. Consider an object projected from the Earth with an initial velocity of 5 km/sec. Find how high this object goes before returning to Earth. Recall that gravity on the surface of Earth is 9.8 m/sec^2 and the radius of the Earth is 6,378 km.

4. A culture of yeast is growing in a limited medium. There are initially 1000 yeast. The culture grows according to the logistic growth equation

$$\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{50,000}\right).$$

Find the general solution to this equation. Determine how long it takes for the original population to double. Also, find how long it takes for the culture to reach half its carrying capacity.

5. a. A reservoir is populated with a new species of game fish. Initially, the game commission releases 1,000 fish. If $P(t)$ is measured in thousands, then $P(0) = 1$. The growth of the fish population satisfies the logistic growth equation

$$\frac{dP}{dt} = 0.3P - 0.001P^2,$$

where t is in years. Solve this differential equation and determine the carrying capacity for this game species in the reservoir. How long does it take to reach 90% of this carrying capacity?

b. When fishing is allowed, the game commission decides to allow 20,000 fish per year. The resulting differential equation with harvesting becomes

$$\frac{dP}{dt} = 0.3P - 0.001P^2 - 20 = -0.001(P^2 - 300P + 20,000).$$

You are given that

$$\frac{1}{P^2 - 300P + 20,000} = \frac{0.01}{P - 200} - \frac{0.01}{P - 100}.$$

With this information and if $P(0) = 300$, then find the solution to the differential equation with harvesting.

c. Find the limiting population for this equation. Sketch a graph of the solution.

6. a. Consider a lake that maintains a constant volume of 500,000 m^3. Suppose that the lake is initially clean, but at $t = 0$ a factory pumps dioxin into a river feeding the lake, making the concentration of dioxin in the river $p = 4\ \mu\text{g/m}^3$. If the river flows into the lake at a rate of $f = 400\ \text{m}^3/\text{day}$, then assuming that the lake is well-mixed, find a differential equation that describes the concentration of dioxin, $c(t)$, in the lake and any time, t. Solve this differential equation.

b. Because of seasonal changes in the flow rates of the river, a more accurate model uses a flow rate of

$$f(t) = 200\,(2 - 2\sin(0.0172t))\,.$$

Use this flow rate to create a modified model for the concentration of dioxin in the lake, then find the solution for this model. (A more realistic model would have fluctuating dioxin concentrations entering, but that results in a non-separable differential equation.)

c. Determine the limiting concentration of dioxin in the lake for both models.

7. Growing cells absorb their nutrients through their surface, which indicates that a volume growth model should be proportional to the 2/3 power. In addition, if a cell is in an environment with competition, the growth rate would decline with time. Thus, a growth law for a growing cell in culture is written

$$\frac{dV}{dt} = k(t)V^{2/3}, \qquad V(0) = 1.$$

a. Suppose that a growth rate is measured to be

$$k(t) = \frac{0.12t}{t^2 + 1},$$

where t is in minutes and includes both the initial lag time of growth and later slowing from competition. Find the maximum growth rate and when it occurs, and sketch a graph of $k(t)$.

b. Solve the differential equation above with the given growth rate, $k(t)$.

c. Find how long it takes for this cell to double in volume.